Florian Cajori.

ALGEBRA FOR SCHOOLS

BY

GEORGE W. EVANS

Instructor in Mathematics in the English High School
Boston, Mass.

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PREFACE.

In the arrangement of topics in this book an effort has been made to preserve the pupil from the besetting sin of conceiving algebraic operations as a species of legerdemain. This end could best be secured, it seemed to me, by making practical problems the point of departure, initially and at each new turn of the subject. With a concrete case in mind, the pupil can hardly fail to perceive not only the need for the process that he is set to study, but also its rational basis and its economy. In this larger appeal to the practical sense it will be found that there is no slighting of mental dexterity, no injurious deviation from accepted methods, and certainly no sacrifice of mathematical rigor.

If the arrangement of chapters here adopted is not acceptable, it is entirely feasible to take the topics in the traditional order. The index will facilitate this rearrangement.

Whether or not teachers agree with me in respect to the ordering of topics or the method of attack which I have found fruitful in class, they are sure to appreciate the very large collection of examples—some thirty-five hundred all told—which are not reprinted from other text-books.

In the following features also I think the book possesses advantages:

1. The careful classification of problems, so that due emphasis may be given to the several types of equations arising from them.

2. The insistence upon a scheduled explanation of steps in the reduction of equations; and the clearness and brevity of reference and explanation obtained by denoting an equation by its serial number enclosed in a circle.

3. The introduction of supplementary sets of constants for some important problems, as on pages 28 and 39 and in Chapter V.

4. The thorough study of literal equations and generalized problems; and especially of literal quadratics.

5. The prominence given to the solution of equations by factoring *as a fundamental method;* and the treatment of "completing the square" as a method of factoring.

6. The treatment of Rules and Formulæ in Chapter II, and of Theorems and Identities in Chapter IV.

7. The separation of Elimination into two parts suitable for linear systems and for linear-quadratic pairs respectively; the discussion of simultaneous, independent, and consistent equations; the "Equations of the New Set," page 160.

8. The treatment of H. C. F. by elimination of highest and lowest terms.

Much of the book has been used, in manuscript and in proof, for class-work, but I can hardly hope that all mistakes have been corrected: notice of remaining errors will be gratefully received.

For friendly aid and suggestion I am indebted to several of my colleagues; and especially to Mr. P. F. Gartland, who scrutinized for me every letter of the proofs.

G. W. E.

English High School, Boston,
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CONTENTS.

ALGEBRA.

CHAPTER I.

THE FIRST USE OF ALGEBRA.

1. Algebra is a method of abbreviating the explanations of problems in arithmetic. It is also used to abbreviate the statement of rules and the demonstration of theorems.

2. This chapter illustrates the first use of Algebra: for abbreviating the explanation of problems in arithmetic. Some problems are here given as examples; the answers to many of them are evident at a glance. The pupil must remember that he is not expected to shorten the arithmetical work, for he cannot do that. He is to put down the explanation briefly and systematically. When that is learned he will find that he can do complicated problems with greater ease, because he can put down in black and white as he goes along the successive steps of his reasoning.

3. Model A.—A father is 6 times as old as his son, and their united ages are 42 years; find the age of each.

Explanation.

(1) The father's age $+$ the son's age $=$ 42 years.
(2) The father's age $= 6 \times$ the son's age.
(3) $6 \times$ the son's age $+$ the son's age $=$ 42 years.
(4) $7 \times$ the son's age $=$ 42 years.
(5) The son's age $=$ 6 years.
(6) The father's age $=$ 36 years.

Abbreviated Explanation.

Let s stand for the number of years in the son's age; then $6 \times s$ will stand for the number of years in the father's age.

$$
\begin{aligned}
&(1) & 6 \times s + s &= 42 \\
&(2) & 7 \times s &= 42 \\
&(3) & s &= 6 \\
&(4) & 6 \times s &= 36
\end{aligned}
$$

EXERCISE I.

1. John is twice as old as Henry, and the sum of their ages is 21 years. Find the age of each.

2. John and Will wish to divide 36 cents so that John shall have twice as much as Will. How much will each have?

3. Paid \$220 for a horse and saddle, and the horse cost 10 times as much as the saddle. Cost of each?

4. A sidewalk laid with a flagstone and a curbstone is 6 feet wide; the flagstone is 11 times as wide as the curbstone. Width of the flag and of the curb?

5. The electoral vote for the state of Pennsylvania is 5 times that for the state of West Virginia. Both count 36 electoral votes. Each counts how many?

6. There are 7 times as many sheep as lambs in a pasture, and in all there are 96. Number of each?

7. Pole 9 feet long partly peeled of bark; bark part twice as long as bare part. Find length of each part.

8. It takes a man 4 times as long to run the first three-quarters of a measured mile as to run the last quarter; he goes the whole mile in 6 minutes. How long for the last quarter?

9. Mercury weighs 13 times as much as water; a quart of mercury and a quart of water weigh in all 28 pounds. Weight of each?

10. A bag of oats, hung on one end of a steelyard beam, weighs 20 times as much as the iron weight attached to the other end of the beam; the total downward pull on the steelyard is $52\frac{1}{2}$ pounds. Find the weight of the bag of oats.

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4. **An equation** is a statement that two things are equal.

5. The algebraic abbreviation for any quantity, or for any combination of quantities, is called an algebraic **expression.**

6. It takes two algebraic expressions to form an equation. The two expressions said to be equal are called, respectively, the first, or left-hand **member**, and the second, or right-hand member, of the equation.

7. The parts of an algebraic expression separated by the signs $+$ and $-$ are called the **terms** of the expression.

8. In order to fix the student's mind on the nature of the processes by which his equations are obtained, it is well for him to indicate beside each equation how it arose from the preceding equations. The additional trouble is slight and the scheduled "explanation" is logically complete.

9. **Model B.**—In a certain collection of insects there are twice as many butterflies as moths; of both there are 39. How many of each ?

Let $x =$ the number of moths.

Then $2x$ * $=$ the number of butterflies.

①	$2x + x = 39$	
②	$3x = 39$	same as ①
③	$x = 13$	② $\div 3$
④	$2x = 26$	③ $\times 2$

Ans. 13 moths; 26 butterflies.

* When two quantities are written together without any sign between them, multiplication is indicated. For representing in any prob-

EXERCISE II.

1. One man is twice as heavy as another, and both weigh 339 pounds. Weight of each?

2. The number of cows and sheep in a certain farmyard is 75, and there are 4 times as many sheep as cows. Number of each?

3. The doctor has twice as many books as the minister, and they both have 2100. How many books has each?

4. In the dog-show are 36 St. Bernards, twice as many long-haired as short-haired. Number of each kind?

5. A locomotive weighs 6 times as much as a car, and both weigh 14 tons. Weight of each?

6. A man is twice as heavy as a boy, and both weigh 200 pounds. Weight of each?

7. A's house cost $2\frac{1}{2}$ times as much as B's house, and both cost $7000. Cost of each?

8. Paid $100 for two lots of tiles; one lot cost 4 times as much as the other. Cost of each lot?

9. Walked 4 times as far in the afternoon as I did in the forenoon; all day together I walked 15 miles. How far in the forenoon?

10. It costs 4 times as much to go from New York to Chicago as from Boston to New York; from Boston to Chicago it costs $25. How much from Boston to New York?

11. The president of a stock company owns twice as many shares in it as his brother; and both own 165. How many does each own?

12. A man had a big bill in his pocket, and asked the price of a house and barn; he found that the house cost

lem numbers of which the values are not given, the latter letters of the alphabet are generally used, most often x. These are matters of universal agreement among writers and students of Algebra (conventions).

7 times the value of the bill, and the barn 3 times the value of the bill, both coming to \$5000. Find cost of house and of barn.

13. Three times as many barrels in one cellar as in another; in the store above half as many as in both cellars. In the whole building 216 barrels. How many in each part?

14. A fortress is garrisoned by 5200 men; and there are 9 times as many infantry and 3 times as many artillery as cavalry. How many are there of each?

Valuation Problems.

10. Model C.—Settled an account of \$48 with \$2 and \$5 bills, using twice as many 5's as 2's. Find the number of bills of each denomination.

Let $x =$ the number of \$2 bills.

Then $2x =$ the number of \$5 bills.

(1)	$10x + 2x = 48$ *	
(2)	$12x = 48$	same as (1)
(3)	$x = 4$	(2) $\div$ 12
(4)	$2x = 8$	(3) $\times$ 2

Ans. Four \$2 bills; eight \$5 bills.

EXERCISE III.

How many bills of each kind must I use to settle the following accounts in the manner described for each account?

1. To pay \$88 I use 5's and 2's only, 3 times as many 2's as 5's.

2. To pay \$72 I use 5's and 2's only, 2 times as many 5's as 2's.

* The statement is about the number of dollars in the values of the bills, not about the number of bills; the value of the \$5 bills is 5 times their number, $5 \times 2x$ or $10x$.

3. To pay \$42 I use 10's and 2's only, twice as many 2's as 10's.

4. To pay \$80 I use 10's and 5's only, twice as many 5's as 10's.

5. To pay \$26 I use 10's and 1's only, 3 times as many 1's as 10's.

6. To pay \$45 I use 2's and 1's only, 7 times as many 2's as 1's.

7. To pay \$78 I use 5's and 1's only, 5 times as many 5's as 1's.

8. To pay \$84 I use 10's and 2's only, 4 times as many 10's as 2's.

9. To pay \$75 I use 10's and 5's only, 7 times as many 10's as 5's.

10. To pay \$183 I use 10's and 1's only, 6 times as many 10's as 1's.

11. Five horses and four donkeys weigh all together 9600 pounds. Each horse weighs 4 times as much as a donkey. Find the weight of each animal.

12. Bought apples for 39 cents; twice as many red ones for 5 cents apiece as green ones for 3 cents apiece. Number of each?

13. Engineers examined a bridge that broke down under the weight of a crowd of people, and decided that the breaking strain was 36 tons. Supposing that there were 4 times as many men as women, the average weight being 150 pounds for a man and 120 pounds for a woman, what was the number of each?

14. Tea of first quality at 60 cents per pound, and 3 times as much tea of second quality at 45 cents per pound, came to \$13.65. Number of pounds of each?

15. Mules at \$40 apiece, and 7 times as many horses at \$125 apiece, cost \$2745. Number of animals in the drove?

16. Drove westward in a buggy for 10 hours, then rode

in a train 5 times as fast for two days and nights; the total journey was 1500 miles. How fast did the buggy go?

17. Lead weighs four times as much as marble; three leaden globes and two marble globes, all of the same size, weigh 56 pounds. Weight of each globe?

18. A team is made up of oxen and mules, one of each in a pair; supposing that an ox can pull 4 times as much as a mule, and the whole team is used to pull 30 tons, how much of this load do the oxen pull?

19. A team consisting of 2 oxen and 5 mules, in which each ox pulls twice as much as a mule, is used to raise a safe weighing 18 tons. What would be the strain on each trace of each mule? What would be the strain on the ox-chain?

20. A horizontal iron beam weighing one ton is suspended by 3 chains, 2 at one end and 1 at the other. What is the pull on each chain?

21. Paid 91 cents with dimes and three-cent pieces, of each the same number. Number of each?

22. Three times as many nickels as two-cent pieces in my purse; in all 51 cents. Number of each?

23. Paid $1.80 with quarters, dimes, and nickels; twice as many dimes and 3 times as many nickels as quarters. Number of each?

24. Bought twice as much coffee at 33 cents as tea at 54 cents, and paid in all $2.40. How many pounds of each were bought?

25. Bought horses at $120, sheep at $14, and chickens at 50 cents; 3 times as many sheep as horses, 7 times as many chickens as sheep. All cost $517.50. How many of each?

26. Two travellers started towards each other from opposite ends of a straight road 132 miles long; one went 4 miles an hour, the other 7. How long before they met?

27. The crew of a towboat consists of an engineer, 2 firemen, 3 deck-hands, and a cook, whose wages are respectively \$2, \$1.50, \$1, and \$1.20 per day. For a certain voyage the pay-roll was \$73.60. How many days out?

28. In a certain mill are employed men at an average wage of \$1.17 per day, 5 times as many women at an average wage of 63 cents per day, and twice as many children (as men) at an average wage of 31 cents per day. The weekly pay-roll of this mill is \$296.40. How many men, women, and children are employed in it?

29. Three families, with 8, 5, and 3 members respectively, divide among themselves a square mile of land, so that there are equal shares for all the individuals. What is the share of each family?

30. A sidewalk is laid with 3 flags and a curb, the flags being each 7 times as wide as the curb. The width of the sidewalk is 11 feet. Find the width of flag and of curb.

Difference of the Unknown Numbers Given.

11. Model D.—A father is 50 pounds heavier than his son, and both weigh 248 pounds. Weight of each?

Let $x =$ the number of pounds the son weighs.

Then $x + 50 =$ the number of pounds the father weighs.

(1)	$x + x + 50 = 248$	
(2)	$2x + 50 = 248$	same as (1)
(3)	$2x = 198$	(2) $- 50$
(4)	$x = 99$	(3) $\div 2$
(5)	$x + 50 = 149$	(4) $+ 50$

Ans. Father 149 pounds; son 99 pounds.

EXERCISE IV.

1. Two cannon-balls and a six-pound weight balanced on a scale 50 pounds. What was the weight of each cannon-ball?

2. A marketman sold 11 sheepskins, and then lost \$2 of the purchase-money; he had \$7.90 left. How much did a sheepskin sell for?

3. On a bill of \$181 part of the account is paid with equal numbers of \$10, \$5, and \$2 bills, and a check for \$28 is given for the remainder. How many bills of each kind were used?

4. Two mules and a horse were bought for \$400, and a horse cost \$40 more than a mule. Cost of each animal?

5. Two bucketfuls of water fill a 15-gallon tub all but 5 quarts. What is the capacity of the bucket?

6. A man starts on a journey of 163 miles; he walks 13 hours, rides twice as fast for 20 hours, and finally has to stop 4 miles short of his journey's end. How fast did he walk?

7. A grocer mixes tea that costs him 20 cents a pound with 3 times as much tea that costs him 38 cents a pound, and sells the mixture for \$10, clearing a profit of 62 cents. Number of pounds of each?

8. My farm is three times as large as the one next to it, and both together, with a ten-acre lot across the road, just make a quarter-section—160 acres. Size of each farm?

9. Two trains start from opposite ends of the same 13-mile track, one going 20 miles an hour and the other 24. They stop when 2 miles apart. How long were they going?

10. Quarters in one pile, twice as many dimes in another, and \$1.13 besides—in all \$6.08. How many quarters and dimes?

11. Bought peas at 59 cents a peck, and 1 peck more of beans at 30 cents a peck; paid 25 cents for lettuce—in all \$5. How many pecks of peas and of beans?

12. Bought sugar at 5 cents, 3 times as much coffee at

50 cents, and paid $1.83 for a ham. The whole bill was $12.68. How much coffee?

13. Sold a house; then sold 5 more at double that price; then sold 35 tons of coal at $5 per ton. Received all together $64,448. Price of first house sold?

14. Three times as many nickels as half-dollars; in all $4.55. Number of each?

15. Three times as many dimes as nickels, 5 times as many three-cent pieces as dimes; in all $5.60. Number of each?

16. Three times as many dimes as dollars, 4 times as many cents as dimes, 5 times as many nickels as cents; and $17.68 all together. Number of nickels?

17. I have 10 cents more than my uncle, and we both have $2.90. How much has each?

18. Suppose each of the 36 boys in a class has the same sum, and the teacher has 15 cents more than all of them together. All the money counts up $79.35. How much has the teacher?

19. Three times as many 10's as 2's, and $7.35 besides, make $167.35. Number of 2's?

20. Seven times as many 1's as 5's, and $1.45 besides, make $37.45. Number of 1's?

21. Three times as many 2's as 5's, 4 times as many 10's as 2's, and $11.11 besides, make $666.11. Number of 2's?

22. Rode 6 miles an hour, then walked 7 times as long at 4 miles per hour, then went 23 miles by train; 91 miles in all. How long did I walk?

23. Made a mixture of water, 3 times as much wine at $1.20 per quart, twice as much bark extract as wine at 75 cents per quart, and one-fourth of a pint of beef extract costing $3.25. Sold it all at a profit of $5 and received $24.45. Number of quarts of bark extract?

24. Five times as many nickels as two-cent pieces, 5

times as many dimes as nickels, 7 times as many quarters as nickels, twice as many dollars as dimes, and a check for \$53.92, making \$300. Number of dollars?

REDUCTION OF EQUATIONS.

12. Consider the problem: A is twice as old as B; 22 years ago he was 3 times as old as B. What are their ages now?

Let $x =$ B's age now. Twenty-two years ago A's age was $2x - 22$ and B's age was $x - 22$. So the equation is $2x - 22 = 3(x - 22)$.

Expressions like $3(x - 22)$ are new to the pupil.

Suppose a farmer contracts to deliver 3 bushels of oats and 5 bushels of barley every day. To get the number of bushels of grain delivered in several days he multiplies not only the quantity of oats but also the quantity of barley by the number of days.

Suppose a farmer receives daily x bushels of grain, and delivers daily 3 bushels. The increase of grain in his store for, say, 7 days would be found thus:

x bushels daily for 7 days, ... received $7x$;
3 bushels daily for 7 days, ... delivered 21.

Subtracting the amount delivered from the amount received, $7x - 21$ is what the daily increase of $x - 3$ amounts to in 7 days.

Consideration of similar cases will make the following principle evident.

13. Whenever an expression of two or more terms is multiplied, EACH TERM of that expression SEPARATELY must be multiplied.

14. The equation $2x - 22 = 3(x - 22)$ now becomes $2x - 22 = 3x - 66$, the only change being a change of form, not of value.

This equation $2x - 22 = 3x - 66$ differs from those we

have been used to in that it has negative terms on each side of the equation, that is, terms to be subtracted.

The left side is, not equal to $2x$, but just 22 short of it. In the same way the right side is 66 short of $3x$.

If 22 is added to each side of the equation, the shortage on the left side will disappear, but the right side will still be 44 short of $3x$. That is, the equation will be $2x = 3x -$ 44. This shortage also disappears when 44 is added to each side, giving $2x + 44 = 3x$, an equation like many we have been solving.

But if 66 is added at first the entire shortage on the right is made up; and on the left it only takes 22 out of the 66 to make up that shortage, leaving 44 to be added to the complete value of $2x$. The shortest way, then, when there are two similar shortages, is to add the larger.

Shortages and Multiplications.

15. Model E.—A is twice as old as B; 22 years ago he was three times as old as B. What are their ages now?

Let $x =$ B's age now; then A's age $= 2x$.

(1)	$2x - 22 = 3(x - 22)$	
(2)	$2x - 22 = 3x - 66$, same as (1)	
(3)	$2x + 44 = 3x$	(2) $+ 66$
(4)	$44 = x$	(3) $- 2x$
(5)	$88 = 2x$	(4) $\times 2$

Ans. A's age 88; B's age 44.

EXERCISE V.

1. A is 5 times as old as B, and 5 years hence will be only 3 times as old as B. Ages now?

2. Eleven years ago A was 4 times as old as B and in 13 years from now he will be only twice as old. Ages now?

3. Rice costs 2 cents a pound more than sugar; 3 pounds

of sugar and 10 pounds of rice come to \$2.40. Cost of each?

4. I can walk 3 miles more in a forenoon than in an afternoon; and between Monday noon and Friday night I can walk 75 miles. How far can I walk in one afternoon?

5. A wall $13\frac{1}{4}$ feet high is built with seven courses of foundation-stone and 41 courses of brick; the courses of stone are 9 inches thicker than the brick. What is the thickness of the foundation-stones?

6. A wall is laid with bricks of two thicknesses, 3 inches and 5 inches; there are three more courses of the thicker kind than of the thinner, and the wall is 9 feet 3 inches high. Number of courses of each?

7. Paid \$2.20 in quarters and nickels; two more nickels than quarters. Number of each kind of coin?

8. A boy starts from his room in college at noon one day to walk to his home, 41 miles off; at two o'clock of the same day the father starts on horseback for the college, riding 3 miles per hour faster than the son can walk; they meet at 5 P.M. How fast can the boy walk?

9. A pile of 27 cannon-balls, in three sizes, weighs 254 pounds; 10 of them are twice as heavy as the 13 lightest; and the others are each by 2 pounds the heaviest in the pile. Find the weight of each size.

10. A tank holding 5940 gallons is filled in 3 hours by three pipes, the first of which carries twice as much, and the other 3 gallons less, per minute, than the third. Number of gallons per minute through each pipe?

11. A is three times as old as B; 15 years ago he was five times as old as B. What are their ages now?

Find the number of coins of each denomination used in the following payments:

12. Paid 39 cents with three-cent pieces and five-cent pieces; 5 more 3's than 5's.

13. Paid 85 cents with nickels and dimes; 2 more nickels than dimes.

14. Paid \$2.05 with quarters and nickels; 1 fewer nickels than quarters.

15. Paid \$4.75 with quarters and halves; 5 fewer quarters than halves.

16. Paid \$1.45 with quarters and dimes; 4 more dimes than quarters.

17. Paid 68 cents with dimes and two-cent pieces; 10 more 2's than dimes.

18. Paid \$2.30 with quarters and three-cent pieces; 2 more 3's than quarters.

19. Paid \$1.09 with two-cent pieces and three-cent pieces; 3 more 3's than 2's.

20. Paid \$1.01 with dimes and three-cent pieces; 12 more 3's than dimes.

21. Paid \$6.40 with nickels and halves; 4 more halves than nickels.

REVIEW.

I. What is the use of Algebra? What is an equation? What is a term?

II. State the first and second members, and the first, second, and third terms, in the following equations:

1. $3x - 5 + 10x = 7 - x.$

2. $x - y + 5 = 3y + a.$

3. $x - x + 1 + 5x - y + 6 = x + 5x + 1.$

4. $4x = 3 + 5x - y + 6.$

III. Solve the following problems:

1. Thirty horses and 40 mules weigh 54 tons; on an average, each horse weighs 100 pounds more than a mule. Average weight of the two kinds of animals?

2. There are in a purse three times as many nickels as dimes, and in all \$1.50. How many nickels?

3. Four cows, 3 calves, and 10 sheep cost $168; a cow costs five times as much as a calf, and a calf costs twice as much as a sheep. Cost of each?

*4.** A man wishes to pay $35 with dollars, halves, and quarters, of each an equal number. Number of each?

5. A grocer mixes tea that cost him 25 cents a pound with four times as much tea that cost him 30 cents a pound, and sells the mixture for $16, clearing a profit of $1.50. Number of pounds of each?

Rule for Solving Simple Equations.

16. An Axiom is a general statement not requiring proof. Examples of axioms are:

17. IF EQUAL QUANTITIES ARE INCREASED OR DIMINISHED BY THE SAME AMOUNT, THEY REMAIN EQUAL.

18. IF EQUAL QUANTITIES ARE MULTIPLIED OR DIVIDED BY THE SAME AMOUNT, THEY REMAIN EQUAL.

19. *Upon these two axioms is based the following rule for solving simple equations:*

(1) Perform any indicated multiplications and unite similar terms.

(2) If there are any terms with − before them, add to each member the quantities lacking.

(3) Subtract from each member the smaller unknown term.

(4) Subtract from each member the known term that stands beside an unknown term.

(5) Divide each member by the coefficient of x.

20. Where a quantity may be separated into two factors, one of these is called the **coefficient** of the other; but by the coefficient of a term is generally meant its numerical factor.

21. Similar terms are those that have the same letters as factors. Similar terms may be united; e.g.,

* Reduce all sums to the denomination of quarters.

$$3x + 4x = 7x$$
$$3x + 4x - 2x = 5x$$
$$3x + 5x + 5 + 2x + 2 = 10x + 7$$
$$7x + 5 - 2x + 3 - x - 4 = 4x + 4$$

Terms not similar cannot be united; e.g.,

$3x + 4y$ is not equal to $7y$ nor to $7x$; it cannot be simplified.

EXERCISE VI.

Perform indicated multiplications:

1. $3(x + 5)$.
2. $5(x + 4)$.
3. $7(2x - 3)$.
4. $13(1 - x)$.
5. $11(3 - 4x)$.
6. $8(4 + 3x)$.
7. $0(3x - 17)$.
8. $8(19 + 5x)$.
9. $201(10 - 71x)$.
10. $3(x - 5) + 5(3 - x)$.

Unite similar terms:

11. $x + y + 7x + 13y + 5 + x$.
12. $3x - 5 + 4x + 2 - 5x + 7 - x$.
13. $7 - 7x + 3 - x - 11 + 9x + x - 1$.
14. $3 + 5x - 8x - 2 + 7x - 10 + x + 15$.
15. $x + y + 1 + x - y + 1 + x + y - 1 - x + y + 1$.
16. $13x - 2x + 17 - 5x - 11 - x + 7x - 23$.
17. $1 + x + 6 - 3x - 10 + 7x - 5x + 4$.
18. $4 + x + 4x + 20 - 3x - 8 + 2x$.
19. $208 - 29x - 191 + 17x - 17 + 53x$.
20. $1089x - 2001 + x + 1987 - 787x - 400 + 350x$.

Perform indicated multiplications and unite similar terms:

21. $3(4 - x) + 7(x - 3) + 5x - 4$.
22. $5(1 - x) + 13(x - 3) + 11x - 19$.
23. $x + 5 + 3(2x - 1) + 7 + 2(1 - 5x) + 10x$.
24. $3 + 4(x - 5) + 3x + 7 + 13(4 - 3x) + 11$.
25. $4(x - 5) + 9(3x - 2) + x + 4(1 - 5x)$.
26. $5(7x - 2) + 13 - 5x + 12(3 - 2x) + 5$.
27. $17(x + 5) + 50(3 - x) + 40x - 10 + 3(x + 1)$.
28. $40(2x - 1) + 45(2 - x) + 17 + 17(x + 17)$.

29. $30(2x + 3) - 15x + 5 + 5(4 - 5x) + 23.$

30. $100(x + 100) - 10x - 10 + 10(2 - x) - 90x.$

Find the value of x in the following equations:

31. $3(x - 7) = 2x - 11.$

32. $2(x + 5) = 10x - 6.$

33. $2x + 5 = 3x.$

34. $5x - 7 = 3(x + 1) - 4.$

35. $10x + 13 = 3(x + 9).$

36. $7(x - 2) = 4(x + 4) + 3.$

37. $x + 5 = 7(x - 10) - x.$

38. $8x - 1 = 5x + 41.$

39. $7(x + 2) = x + 32.$

40. $5(1 - x) = 1 - 3x.$

41. $5(x + 4) = 14x + 2.$

42. $3(x + 3) = 3(3x + 1) - 12.$

43. $4(x + 5) = 6x + 6.$

44. $5(x + 1) = 6(3 - x) - 2.$

45. $2(x - 3) = x + 7.$

46. $2x = 5(x - 1) - 2x.$

47. $17(x - 2) = 2(2x + 1) + x.$

48. $13(5 - x) = 3x + 1.$

49. $2x - 2 = x + 4.$

50. $5x - 8 = 6(x - 3).$

51. $3(2x - 5) = 4x + 1.$

52. $25 - 4x = 2(x - 4) - 3.$

53. $7 - x = 3(4 - x) + 1.$

54. $2(x + 5) = 5(5 - x) + x - 3.$

55. $4 + x + 4(5 - x) = 3x + 2(4 - x).$

56. $2x + 5 = 5(x - 2) + 2x - 10.$

57. $1 + x + 3(2 - x) = 7(3 - x).$

58. $13(8 - x) + 2(x - 6) = 2x + 1.$

59. $23(10 - x) - 4(x - 7) = 3x + 3(x - 13).$

60. $9(x + 1) + 5 = 5x + 6(x - 1).$

61. $3(x-4)=4(x-3)-17$.
62. $11(x+3)-100=9(x+5)$.
63. $15(7x-3)=8(13x+7)-20$.
64. $7(19x-21)=11(12x-3)-30$.
65. $9(25x+11)=19(12x-5)+2$.
66. $51x-1135+13(103-x)=11(2x+20)$.
67. $203-17x=3(x-505)-2$.
68. $1170+(x-4)=11(x-3)+22$.
69. $9(103x-820)=206(2-x)+139$.
70. $800(x-10)+2(11x-32\ )-1-35x$.
71. $3000(x-5)=10(100-27x)+355-x$.
72. $10(2x-3)=2(x+15)+6$.
73. $3(3x-11)+2(5x+2)=5+9(2x-5)$.
74. $5(7x-2)+2(5x-13)=3(11x+2)+26$.
75. $17(3x-1)+3(5x-17)=11(5x-3)+x+5$.
76. $13(x-11)+17(2x-27)-1=3(x-29)$.
77. $101(5x+17)=203(3x+23)+3x-3059$.
78. $4(x-13)=13(x-4)-81$.
79. $117(x-25)+25(x+117)=3x+50+3(7-x)$.
80. $8(x-1)+4(9-4x)=4+17(3-x)$.
81. $17(2x+4)+13(2x-5)=3(x+17)$.
82. $51x-1137+13(103-x)=11(2x+20)-2$.
83. $\frac{5}{6}(18x-252)=\frac{2}{3}(15-3x)+1$.
84. $\frac{7}{8}(24x-800)=\frac{2}{5}(45-5x)-5$.
85. $\frac{4}{5}(60x-875)=\frac{1}{7}(35+10x)-\frac{3}{7}x$.
86. $\frac{2}{3}(1185x-5925)=\frac{9}{11}(11x-55)$.
87. $\frac{1}{3}(93x-138)=\frac{2}{5}(10x-25)$.
88. $\frac{4}{7}(56x+42)=5(2x+16)$.
89. $\frac{5}{6}(42x-48)-x=3(x+17)$.
90. $\frac{3}{2}(60-22x)+\frac{9}{10}(30-50x)=0$.
91. $\frac{1}{5}(5x+25)+\frac{3}{4}(4x-28)=\frac{2}{3}(3x+9)$.
92. $\frac{2}{3}(21x-57)=2(6x+17)$.
93. $13x+3+\frac{5}{6}(18-12x)=3(7-x)+11-x$.
94. $31(x-22)+\frac{2}{3}(21x+18)=2x+883$.

Sum of the Unknown Numbers Given.

22. Model F.—A merchant has grain worth 9 cents per peck, and other grain worth 13 cents per peck; in what proportion must he mix 40 bushels so that the mixture may be worth 40 cents per bushel?

$$x = \text{number of bushels at 36 cents.}$$
$$40 - x = \text{number of bushels at 52 cents.}$$

(1)	$36x + 52(40 - x) = 1600$	
(2)	$36x + 2080 - 52x = 1600$	same as (1)
(3)	$2080 - 16x = 1600$	same as (2)
(4)	$2080 = 1600 + 16x$	(3) $+ 16x$
(5)	$480 = 16x$	(4) $- 1600$
(6)	$30 = x$	(5) $\div 16$

Ans. 30 bushels of the cheaper and 10 of the dearer.

EXERCISE VII.

1. Twenty coins, quarters and halves, came to \$5.75. Number of each?

2. \$4.04 in dollars, dimes, and cents; 44 coins in all; 9 times as many dimes as dollars. Number of each?

3. 108 coins, dimes and cents, amount to \$4.32. Number of each kind?

4. Fourteen coins, dimes and three-cent pieces, came to 77 cents. Number of each kind?

5. A bridge broke down under a strain of 28,500 pounds, caused by a crowd of 200 people. The average weight of a man being 150 pounds, and of a woman 120 pounds, how many men were there in this crowd?

6. Bought 30 pounds of sugar of two sorts for \$2.27; the better cost 10 cents per pound, and the poorer 7 cents. Number of pounds of each sort?

7. Bought 15 apples for 59 cents; red ones at 5 cents, green ones at 3 cents. Number of each?

8. Find four consecutive numbers whose sum is 94.

9. A grocer is offered \$15 for 50 pounds of tea, and mixes 25-cent tea with 35-cent tea so as to make \$1 on the transaction. How many pounds of each grade of tea in the mixture?

10. A merchant has grain worth 11 cents per peck, and other grain worth 15 cents per peck; in what proportion must he mix 20 bushels so that the mixture may be worth 48 cents per bushel?

11. \$1.97 in two-cent pieces, three-cent pieces, and dimes; 3 times as many three-cent pieces as two-cent pieces; 40 coins in all. Number of each?

12. A man starts on a wager to walk 2000 miles in 50 days; after travelling for 30 days he finds that he can go 5 miles slower per day and still come out 20 miles ahead. How fast did he go at the start?

13. One hundred coins, quarters and dimes, amount to \$20.20. Number of each?

REVIEW.

I. Find the value of the unknown number represented by the letter x in each of the following equations:

1. $5(x-7)=3x-9$.

2. $7(x-3)=5x-3$.

3. $4(x+5)=6x-20$.

4. $3(x+2)=4x-5$.

5. $6(x-3)=5(x-2)$.

6. $8(x-3)=5x-3$.

7. $5(x+4)=7x-4$.

8. $11(x+1)=13x+1$.

9. $13(x-10)=3x$.

10. $9(x+6)=10(2x+1)$.

11. $11(x+3)=10(x+1)+1$.

12. $8(x+10)=12(x+7)-8$.

13. $100(x-5)=60(x+3)$.

14. $4(x+5)=2(3x-10)$.

15. $3(x-10)+2(x+4)=6x-22$.

16. $8(10-x)=5(x+3)$.

17. $3x+4(x-2)=1+3(2x-3)$.

18. $19(x-2)=6(3x-5)$.

19. $3(3x-5)=4(x+5)$.

20. $15(5x-3)=12(4x+3)$.

II. Solve the following problems:

1. Borrowed some coins, and paid back 4 fewer coins of another sort; the coins borrowed were quarters, those repaid halves; paid 25 cents too much.

2. Borrowed some coins, and paid back 5 fewer coins of another sort; the coins borrowed were quarters, those repaid were halves; still owed 50 cents.

3. Borrowed some coins, and paid back 33 more coins of another sort; the coins borrowed were halves, those repaid were nickels; still owed $1.50.

4. Borrowed some coins, and paid back 52 less coins of another sort; the coins borrowed were three-cent pieces, those repaid were quarters; paid $1.30 too much.

5. Borrowed some coins, and paid back 3 less coins of another sort; the coins borrowed were two-cent pieces, those repaid were dimes; paid 74 cents too much.

6. Lost a few quarters, then found 5 more coins than I lost, only these coins were the old-fashioned twenty-cent pieces; on the whole I was the gainer by 65 cents. How many quarters did I lose?

III. What is the difference in treatment between an equation and an expression?

IV. What important principle of multiplication is illustrated in the solution of the equations in I?

V. Give the exact meaning of the word coefficient; also the sense in which it is generally used.

EXERCISE VIII.

Find the ages of the persons described in the following problems:

1. A is 12 years older than B; in 2 years he will be 4 times as old as B.

2. A is 6 times as old as B; in 4 years he will be only 4 times as old as B.

3. A is 18 years older than B; in 5 years he will be 3 times as old as B.

4. A is 9 times as old as B; in 6 years he will be only 6 times as old as B.

5. A is 7 years younger than B; in 4 years he will be half as old as B.

6. In 3 years A will be $\frac{1}{5}$ as old as B; now he is 20 years younger than B.

7. A's age at present is $\frac{1}{3}$ of B's; 10 years ago it was $\frac{1}{7}$ of B's age.

8. A was 8 times as old as B one year ago; he is now only 5 times as old as B.

9. A is 30 years older than B; 5 years hence his age will be 6 times B's.

10. A is 7 times as old as B; in 10 years he will be only twice as old as B.

The Problem of the Digits.

23. Model G.—In a number of two digits, the first digit is double the second; and if 27 be subtracted from the number, the digits are reversed. Find the number.

Let $x =$ the units figure; then $2x =$ the tens figure.

The value of the number is $10 \times 2x + x$.

(1)	$20x + x - 27 = 10x + 2x$	
(2)	$21x - 27 = 12x$	same as (1)
(3)	$21x = 12x + 27$	(2) $+ 27$
(4)	$9x = 27$	(3) $- 12x$
(5)	$x = 3$	(4) $\div 9$

Ans. 63.

EXERCISE IX.

1. In a number of two digits the first digit is 3 times the second; and if 36 is subtracted from the number, the digits are reversed. Find the number.

2. In a number of two digits the second number is 3 times the first; adding 54 reverses the digits. Find the number.

3. In a number of two digits the first digit is 4 times the second; subtracting 54 reverses the digits. Find the number.

4. In a number of two digits the first digit is twice the second; subtracting 36 reverses the digits. Find the number.

5. The sum of the two digits of a number is 11; and if 27 be added to the number, the digits are reversed. Find the number.

6. The first digit of a number less than 100 is 1 less than double the second; and if 18 be taken from the number, the digits are reversed. Find the number.

7. The sum of the two digits of a number is 9; and if 45 be added to the number, the digits are reversed. Find the number.

8. The difference of the two digits of a number is 1 less than the units figure; and if 18 be taken from the number, the digits are reversed.* Find the number.

9. The sum of the two digits of a number is 11; and subtracting 45 reverses the digits. Find the number.

10. The smaller digit of a number is 5 less than 3 times the larger; and if 9 be added to the number, the digits are reversed. Find the number.

11. The difference of the two digits of a number is 1 less than 3 times the smaller; and if 45 be added to the number, the digits are reversed. Find the number.

12. The sum of the digits of a number is 11 less than 3 times the larger; and if 27 be added to the number, the digits are reversed. Find the number.

* The fact that subtraction reverses the digits shows which digit is the less.

Numbers in Other Scales.

24. Model H.—The sum of the two digits of a number is 7; in the scale of 6 the number would be 16 less. Find the number.

Let $x =$ the figure in the tens place; then $7 - x =$ the figure in the units place.

$10x + 7 - x =$ the value of the number in the ordinary or decimal scale;

$6x + 7 - x =$ the value of the number in the scale of 6.

1	$10x + 7 - x = 16 + 6x + 7 - x$	
2	$9x + 7 = 5x + 23$	same as 1
3	$4x = 16$	2 $- 7 - 5x$
4	$x = 4$	3 $\div 4$
5	$7 - x = 3$	

Ans. 43.

EXERCISE X.

1. The first digit of a number exceeds the second by 5; and if the number were in the scale of 7 instead of the decimal scale of notation, its value would be 21 less. Find the number.

2. The first digit of a number is less than the second by 3; and if the number were in the scale of 12 instead of the decimal scale of notation, its value would be 4 more. Find the number.

3. The sum of the two digits of a number is 9; and if the number were in the scale of 11, its value would be 7 more. Find the number.

4. The sum of the two digits of a number is 13; and if the number were written in the scale of 8, its value would be 12 less. Find the number.

5. The sum of the two digits of a number is 10; if

written in the scale of 8, the number would be 6 less. Find the number.

6. The first digit of a number is twice the second; if written in the scale of 7, the number would be 18 less. Find the number.

7. The first digit of a number is 3 times the second; if written in the scale of 11, the number would be 9 more. Find the number.

8. The first digit of a number is 1 more than twice the second; if written in the scale of 8, the number would be 14 less. Find the number.

9. The second digit of a number is 1 less than 3 times the first; in the scale of 11 the number would be 3 more. Find the number.

10. The sum of the two digits of a number is 1 more than twice the second; in the scale of 13 its value would be 12 more. Find the number.

11. The sum of the two digits of a number is 17; in the scale of 7 its value would be 27 less. Find the number.

12. The first digit of a number exceeds the second by 2; if written in the scale of 8, the number would be 14 less. Find the number.

12. The sum of the two digits of a number is 7; in the scale of 6 the number would be 20 less. Find the number.

14. The two digits of a number are equal; in the scale of 11 the number would be 1 more. Find the number.

Current Problems.

25. Model I.—A man who can row 5 miles per hour in still water finds that it takes him 5 hours to row up-stream to a point from which he can return in 4 hours. How fast does the current flow?

Let $x =$ number of miles per hour the current flows;

Then $5 + x =$ number of miles per hour the man can row down-stream,

And $5 - x =$ number of miles per hour the man can row up-stream.

①	$5(5 - x) = 4(5 + x)$	
②	$25 - 5x = 20 + 4x$	same as ①
③	$25 = 20 + 9x$	② $+ 5x$
④	$5 = 9x$	③ $- 20$
⑤	$\frac{5}{9} = x$	④ $\div 9$

Ans. River flows $\frac{5}{9}$ of a mile per hour.

EXERCISE XI.

1. A river flows 2 miles per hour, and a fisherman finds that he can row up-stream a few miles in 6 hours, but it takes him only 3 hours to come back. How fast can the fisherman row in still water?

2. A man starts to row down the river to the steamboat landing and back in 4 hours; it takes him 2 hours to get down there, and at the end of his time coming back he is 4 miles short of his starting-place. The man can row 3 miles per hour. How fast does the river flow? How far off is the steamboat landing?

3. I lent my steam launch to a friend who did not know that it would run only 10 hours without refiring. He went down the river, which has a 2-mile current, for 6 hours, and on his return trip the launch stopped 30 miles away from the boat-house. What is the speed of the launch?

4. A man who can row 5 miles per hour finds that it takes him 3 times as long to go up a river as to go down; find the speed of the current.

5. The two fire-cisterns in a certain town are of equal size, and are being filled from mains that pour into each

10,000 gallons per hour; but the one near the business part of the town, being considered the case of more urgent need, has a pump rigged from the other cistern so as to hasten its filling. Consequently it is half-full in 7 hours, while the other cistern does not get half-full till 6 hours later. What is the capacity of the pump? Of the cisterns?

6. I had 4 hours to spare, and started for a row down-stream; but I forgot that the current flowed a mile an hour, so I took half my time before I turned to go back; the consequence was that I got home two hours late. How fast can I row in still water?

7. A man can row 4 miles per hour in still water; going up-stream he goes only $\frac{2}{3}$ as fast as he comes down. Find the rate of the stream.

8. A man who can row 6 miles per hour in still water goes 1 mile farther in 3 hours coming down-stream than he does in 4 hours going up. Find the speed of the current.

9. I had 6 hours to spare, and started for a row down-stream; but I forgot that the current flowed $1\frac{1}{2}$ miles per hour, so I took half my time before I turned to go back; the consequence was that I was only half-way home when my time was up. How fast can I row in still water?

10. A centipede in a grain-elevator, crawling on a belt which moves 11 feet per second, takes 10 seconds to go the whole length of the building when he goes with the belt, and 2 minutes to return, against the motion of the belt. How fast could the centipede go if the belt were still?

11. A fish that can swim 15 miles per hour takes 12 minutes to pass up through a certain part of the river, but takes only 10 minutes to come down again. Speed of the current there?

New Work on Old Patterns.

26. In Problem I mark a small letter *a* over the number 5 in "row 5 miles," a letter *b* over the number 5 in "takes him 5 hours," and a letter *c* over the number 4 in "return in 4 hours." Then solve the problem, putting instead of the numbers marked *a, b,* and *c*, the values represented below, each set making a new problem.

	a	*b*	*c*
1.	4	3	1
2.	6	4	2
3.	$5\frac{1}{2}$	8	3
4.	$4\frac{1}{2}$	4	2
5.	$3\frac{1}{2}$	4	3

In example 1 of the preceding exercise, letter the numbers 2, 6, and 3 in the same way, and then use the following sets of values instead,* just as in the case of Problem I:

	a	*b*	*c*
1.	$1\frac{1}{2}$	5	2
2.	1	6	5
3.	3	8	5
4.	3	4	1
5.	$2\frac{1}{2}$	8	3

FRACTIONAL EQUATIONS.

27. When equations arise that contain fractions, it is necessary to multiply both sides of the equation by num-

* For some of the problems suggested here it may prove more reasonable to describe the powers of a fish instead of a fisherman.

bers such as will change the fractions to whole numbers, as in the following examples :

Model J.—① $\frac{x}{3} = 2$

② $x = 6$ ① $\times 3$

Model K.—① $\frac{2y}{3} = \frac{3}{5}$

② $2y = \frac{9}{5}$ ① $\times 3$

③ $10y = 9$ ② $\times 5$

④ $y = \frac{9}{10}$ ③ $\div 10$

One line of the work in example K could have been saved if we had multiplied both sides of ① by 15 at once, instead of first by 3 and then by 5.

Model L.—① $\frac{x}{2} - \frac{x}{3} = \frac{x}{4} - \frac{1}{2}$

② $x - \frac{2x}{3} = \frac{x}{2} - 1$ ① $\times 2$

③ $3x - 2x = \frac{3x}{2} - 3$ ② $\times 3$

④ $x = \frac{3x}{2} - 3$ same as ③

⑤ $2x = 3x - 6$ ④ $\times 2$

⑥ $2x + 6 = 3x$ ⑤ $+ 6$

⑦ $6 = x$ ⑥ $- 2x$

Here if we had multiplied ① by 12, the least common multiple of the denominators, we should have had:

Model L.—① $\frac{x}{2} - \frac{x}{3} = \frac{x}{4} - \frac{1}{2}$

② $6x - 4x = 3x - 6$ ① $\times 12$

③ $2x = 3x - 6$ same as ②

④ $2x + 6 = 3x$ ③ $+ 6$

⑤ $6 = x$ ④ $- 2x$

EXERCISE XII.

Solve the following equations (that is, find the value of x implied by them):

1. $\frac{x}{3}+\frac{x}{5}+3=x-4.$

2. $\frac{x}{2}+\frac{x}{3}+\frac{1}{2}=x-1.$

3. $\frac{x}{3}-\frac{x}{7}=\frac{x}{6}+1.$

4. $\frac{x}{4}+\frac{x}{6}-\frac{1}{3}=\frac{x+2}{3}.$

5. $\frac{x}{2}-\frac{x}{5}=\frac{x}{2}-3.$

6. $\frac{x}{7}-\frac{3x}{4}+17=0.$

7. $-\frac{2}{3}+\frac{x}{6}=\frac{5x-8}{36}.$

8. $\frac{2x-1}{2}-\frac{1}{3}=5+\frac{x}{6}.$

9. $\frac{7x-6}{5}+\frac{x}{3}=\frac{3x-5}{2}+x-1.$

10. $\frac{3x-7}{4}-\frac{3x}{11}=\frac{x-5}{2}.$

28. Model M.—Bought a certain number of picture-cards at 2 for a nickel, and the same number at 3 for a nickel. Sold them all at a uniform price of 2 cents apiece, and lost a quarter. How many did I buy?

Let $x=$ the number of cards of each kind; then $\frac{5x}{2}$ would be the cost of one lot and $\frac{5x}{3}$ of the other.

(1)	$\frac{5x}{2}+\frac{5x}{3}=4x+25$	
(2)	$5x+\frac{10x}{3}=8x+50$	(1) $\times$ 2
(3)	$5x+10x=24x+150$	(2) $\times$ 3
(4)	$25x=24x+150$	same as (3)
(5)	$x=150$	(4) $-$ $24x$

EXERCISE XIII.

1. What number is that $\frac{5}{6}$ of which exceeds 27 by as much as $\frac{1}{12}$ of it is less than 6?

2. A, B, and C shared a sum of money so that A had \$30 more than half, B \$33 more than a fourth, and C the remainder, which was \$16. Find the shares of A and B.

3. A has \$107, B \$45. B gives A a certain sum and then has only $\frac{1}{3}$ as much as he. How much did B give A?

4. If a certain number be successively subtracted from 53 and 62, and then $\frac{4}{5}$ of the first remainder be taken from $\frac{6}{7}$ of the second, the last remainder will be 10. Find the number first subtracted.

5. A and B started eastward from Palmer. At noon A had gone $\frac{3}{5}$ of the distance to Boston, B had gone $\frac{4}{7}$ of it, and they were just 2 miles apart. What was the whole distance to Boston?

6. From a cask of wine $\frac{3}{20}$ had leaked out, $\frac{2}{3}$ had been drawn out, and there were 11 gallons left. How much was there at first?

7. Bought pineapples at the rate of \$4 for 7 dozen, and sold them at the rate of \$2 for 3 dozen, gaining \$4. How many pineapples were there?

8. A man can walk 4 miles an hour or ride 7 miles an hour. How long will it take him to go 66 miles, if he rides half the time and walks half the time?

9. The fore wheel of a carriage has a circumference of 6 feet, and the hind wheel a circumference of 10 feet. When the fore wheel has made 25 turns more than the hind wheel, how far has the carriage gone?

10. A woman had a basket of eggs, and sold $\frac{1}{3}$ of them, with $\frac{1}{3}$ of an egg besides; what she had left were 3 more than half what she had at first. How many eggs had she at first?

11. A boy has an hour for exercise; how far may he ride

with his father, at the rate of 10 miles per hour, before getting out to walk back? He can walk $3\frac{1}{3}$ miles per hour.

REVIEW.

I. What is an axiom? Upon what axiom does the reduction of fractional equations depend?

II. What would express the value of a number whose three figures were represented by x, y, and z? What conclusion could you draw as to the relative value of these letters, if you were told that subtracting 198 reverses the order of the figures?

III. Solve these problems:

1. Paid \$3.75 with halves and quarters; 3 times as many quarters as halves. Number of each?

2. Paid \$2.45 with dimes and nickels; 5 times as many nickels as dimes. Number of each?

3. Paid \$1.36 with three-cent pieces and two-cent pieces; 7 times as many 2's as 3's. Number of each?

4. Paid \$1 with cents and two-cent pieces; 18 times as many cents as two-cent pieces. Number of each?

5. Paid \$2.26 with dimes, two-cent pieces, and halves; 3 more two-cent pieces than dimes; 25 coins in all. Number of each?

6. Paid \$7.50 with nickels, quarters, and halves; 1 more quarter than halves; 55 coins in all. Number of each?

7. Paid \$3.23 with two-cent pieces, three-cent pieces, and dimes; 14 more dimes than two-cent pieces; 49 coins in all. Number of each?

8. Paid \$5.90 with three-cent pieces, dimes, and quarters; 6 fewer dimes than quarters; 85 coins in all. Number of each?

9. Paid \$2.24 with halves, two-cent pieces, and nickels; 5 more nickels than halves; 28 coins in all. Number of each?

10. Paid \$2.66 with cents, quarters, and dimes; 2 more dimes than cents; 29 coins in all. Number of each?

EXERCISE XIV.

Find the value of x in the following equations:

1. $\frac{x-1}{4}-\frac{x}{5}=2x-10.$

2. $\frac{x+1}{2}+\frac{x+3}{4}=x-2.$

3. $\frac{x+3}{2}+\frac{x+5}{3}+5=2x.$

4. $\frac{2x+1}{7}-\frac{4x}{6}=2-x.$

5. $\frac{x+5}{3}+\frac{x-5}{2}=\frac{3x}{4}.$

6. $x+\frac{3x}{2}+\frac{5x-2}{9}=2x+4.$

7. $3x+\frac{x-2}{2}=\frac{3-x}{3}+2x.$

8. $\frac{x-1}{7}+\frac{3x-1}{5}+\frac{1}{7}=\frac{4x-3}{5}+\frac{x-7}{3}.$

9. $3x+5+\frac{x+2}{2}=\frac{5x-1}{3}+5x.$

10. $10x=\frac{3x-1}{2}+\frac{15x-3}{7}+7x-1.$

The Minus Sign before a Fraction.

29. In example 10 of the preceding exercise the equation would have assumed quite a different form if, say, the fraction $\frac{15x-3}{7}$ had been subtracted from each side of the equation.

It would then be

$$\text{(1)}\quad 10x-\frac{15x-3}{7}=\frac{3x-1}{2}+7x-1,$$

and in solving this especial pains must be taken on account of the fact that when we multiply by 14 the fraction

$\frac{15x-3}{7}$ becomes a whole number, and that whole number has to be subtracted from the preceding term of the equation.

$$② \quad 140x - (30x - 6) = 21x - 7 + 98x - 14.$$

Here we have $30x - 6$ to subtract from $140x$.

If we had to take $30x$ from $140x$, the remainder would be $110x$; now, however, since we are taking 6 less than $30x$, our remainder will be 6 more, or $110x + 6$.

The equation is therefore solved as follows:

$$① \quad 10x - \frac{15x-3}{7} = \frac{3x-1}{2} + 7x - 1$$

②	$140x - (30x - 6) = 21x - 7 + 98x - 14$	① $\times$ 14
③	$110x + 6 = 119x - 21$	same as ②
④	$110x + 27 = 119x$	③ $+ 21$
⑤	$27 = 9x$	④ $- 110x$

EXERCISE XV.

1. $\frac{x-1}{2} - \frac{x+3}{5} = \frac{2}{5}$.
2. $\frac{5x-4}{8} - \frac{3x+2}{7} = \frac{x-4}{4}$.
3. $\frac{2x-1}{3} + \frac{x-5}{6} = \frac{x+3}{2}$.
4. $5x - \frac{x+7}{10} = 8\frac{1}{2} + \frac{x+1}{5}$.
5. $x + \frac{x+5}{6} = 3 - \frac{3x+5}{8}$.
6. $\frac{x}{4} - \frac{x-5}{3} = \frac{x-8}{4}$.
7. $\frac{x-4}{2} - \frac{x-5}{8} = \frac{x+5}{7}$.
8. $\frac{x+5}{6} - \frac{x+1}{8} = \frac{x+3}{4}$.
9. $\frac{20-3x}{6} - \frac{10-3x}{3} = \frac{x}{2}$.
10. $\frac{x+7}{15} - \frac{3x+1}{25} = \frac{7x-11}{45} - 1$.

11. $5x + 20 - 2(x + 2) = 40 - 4(x - 15).$

12. $10(x + 1) - (3x + 5) = 30 - 10x + 2(x + 35).$

13. $30(x + 6) - 10(x + 3) = 311 - 3x.$

14. $2x + 3(x + 9) = 200 - 4(90 + 3x).$

15. $5(x - 6) + 17 = 110 - 2(x - 5).$

16. $13(x - 13) + 10 = 200 - 2(75 - x).$

17. $200 - 7(60 - 5x) = 3(2x - 11) - 100.$

18. $50x - 10(x + 10) = 40 + x - 8(2 - x).$

19. $274(2x - 10) - 163(8 + x) - 48x = 0.$

20. $7x - 103(x - 17) + 17(3x - 103) + 135 = 0.$

21. $\frac{11}{2} - 100(13x - 74) = 7(7x - 2).$

22. $\frac{3}{5} - \frac{24}{35}(4x - 25) = \frac{1}{5}(5x - 61).$

23. $2x - 7 - \dfrac{x - 5}{2} = 17 - \dfrac{3x - 31}{4}.$

24. $25 - \dfrac{x + 17}{2} = \dfrac{7x - 2}{3} + \dfrac{8x - 13}{9}.$

25. $\dfrac{10x - 7}{9} + 13 = \dfrac{6x + 33}{5} + \dfrac{3x - 11}{2}.$

26. $\dfrac{4x - 7}{5} - 7 = \dfrac{3x - 11}{4} - \dfrac{x + 17}{6}.$

27. $\dfrac{5(x + 1)}{6} - \left(x - \dfrac{2x - 7}{3}\right) = x - 10.$

28. $\dfrac{11x + 19}{6} - \dfrac{6x - 5}{3} = \dfrac{6x + 1}{6}.$

29. $\frac{2}{7}(3x + 1) - \left(x - \dfrac{x - 1}{2}\right) = \dfrac{x + 1}{3}.$

30. $\dfrac{8x - 3}{7} - \dfrac{4x - 7}{5} = 5x - 13.$

31. $\dfrac{7x - 11}{4} - \dfrac{6x + 5}{5} = \dfrac{9x - 5}{22}.$

32. $\dfrac{2x + 7}{3} - \dfrac{3x + 4}{5} = x - 5.$

33. $\dfrac{x + 3}{5} - \dfrac{2x - 13}{11} = \dfrac{x}{3} - 2.$

Day's Work and Cistern Problems.

30. Model N.—A can do work in 4 days which B would take 12 for. How long for both?

Let $x =$ the number of days for both to do the work.

Then in one day both can do $\frac{1}{x}$ of the work.

A can do $\frac{1}{4}$ of the work.

B can do $\frac{1}{12}$ of the work.

①	$\frac{1}{4}+\frac{1}{12}=\frac{1}{x}$	
②	$\frac{x}{4}+\frac{x}{12}=1$	① $\times\ x$
③	$3x+x=12$	② $\times$ 12
④	$4x=12$	same as ③
⑤	$x=3$	④ $\div$ 4

Ans. Three days for both.

EXERCISE XVI.

1. A can do work in 15 days, B the same in 18 days. How long for both?

2. A vessel can be emptied by three taps; by the first alone in 80 minutes, by the second alone in 200 minutes, and by the third alone in 5 hours. How long if all three are open?

3. A can do a job in $2\frac{1}{3}$ hours which B can do in $1\frac{2}{5}$ hours and C in $3\frac{1}{2}$ hours. How long for all together?

4. A does $\frac{2}{5}$ of a piece of work in 10 days; then he calls in B to help him, and they finish the work in 3 days. How long would B take to do the work by himself?

5. A bath-tub is filled in 40 minutes and emptied by the waste-pipe in 1 hour. How long will it take to fill it with the waste-pipe open?

6. A cistern can be filled in 12 minutes by one pipe alone, or in 8 minutes if the second pipe is also turned on. How long to fill it with the second pipe alone?

7. A cistern holding 2400 gallons can be filled in 15 minutes by three pipes, one of which lets in 10 gallons more, and the other 4 gallons less, per minute, than the third. How many gallons flow through each pipe per minute? *

8. A can do a job by himself in 6 days, B can do it in 10 days, and both together with C to help them in $2\frac{6}{7}$ days. How long for C to do it alone?

9. A cistern can be filled from the water-main in 12 hours; if also an extra pump is rigged from a neighboring cistern, it can be filled in 8 hours; and with all the fire-engines in town to help by pumping water from a lake near by it could be filled in 6 hours. How long would it take to fill the cistern if the main were shut off, and the filling depended entirely on the extra pump? How long if the filling depended entirely on the fire-engines?

10. A, B, and C are working together on a job that any one of them could do alone in 36 days; but when the job is half done C goes on a strike, and, by annoying A and B, hinders the work as much as he could have helped it if he had continued at work. When the job is done how many days must A be paid for? How long did C work? How many days did his annoyance delay the completion of the job?

11. Starting at noon, I can get to town on a wagon at 1:48 o'clock; if I walked, I could get there at 2:15 o'clock. If I agree to walk half the time, what time must I get off the wagon?

* The method of solving this example is quite different from the preceding six. It is introduced here lest the pupil should conclude that all "cistern" examples had to be done on the same plan as Model N.

The Clock Problem.

31. Model 0.—At what time between 4 and 5 o'clock are the hands of a clock 9 minutes apart?

Let $x =$ the number of minutes past 4.

Then $\frac{x}{12} =$ the number of minute-spaces traversed by the hour-hand in x minutes.

If the hour-hand is 9 minutes ahead of the minute-hand, the equation becomes

①	$20 + \frac{x}{12} = x + 9$	
②	$240 + x = 12x + 108$	① $\times 12$
③	$132 = 11x$	② $- x - 100$
④	$12 = x$	③ $\div 11$

Ans. 12 minutes past 4.

If the minute-hand is ahead of the hour-hand, the equation becomes

①	$20 + \frac{x}{12} = x - 9$	
②	$240 + x = 12x - 108$	① $\times 12$
③	$348 = 11x$	② $+ 108 - x$
④	$31\frac{7}{11} = x$	③ $\div 11$

Ans. $28\frac{4}{11}$ minutes of 5.

Of these two answers the first only can be realized on an ordinary clock, because the mechanism of the clock is such that $28\frac{4}{11}$ minutes of 5 is never indicated by the position of the hands. They move by jerks, one for each tick of the pendulum, and pendulums that tick elevenths of a second would not have any reason for existence other than the need of illustrating this problem.

EXERCISE XVII.

Instead of the numbers 4, 5, *and* 9, *in the statement of Model O, use the following sets of numbers, and solve the resulting problems :*

1. 4; 5; 2.	**8.** 8; 9; 7.	**15.** 10; 11; 21.
2. 3; 4; 4.	**9.** 8; 9; 4.	**16.** 12; 1; 22.
3. 6; 7; 3.	**10.** 5; 6; 3.	**17.** 4; 5; 24.
4. 7; 8; 2.	**11.** 12; 1; 27.	**18.** 7; 8; 24.
5. 3; 4; 7.	**12.** 9; 10; 26.	**19.** 9; 10; 23.
6 5; 6; 14.	**13.** 1; 2; 21.	**20.** 1; 2; 17.
7. 6; 7; 14.	**14.** 2; 3; 26.	

21. A student of music always practised at the piano between 5 P.M. and 7 P.M. One day he noticed as he began to practise that the hands of his watch were exactly 3 minutes apart; he practised until they were again 3 minutes apart, and then stopped. How long was he practising?

22. In a mile race on an oval track, 11 laps to the mile, a runner has 200 feet the start of a bicyclist and only moves $\frac{1}{5}$ as fast. Where will the bicyclist pass the runner the first time?

23. Where will the bicyclist pass the runner the second time?

24. If they both start at the starting-post, and go opposite ways, where will they meet the second time?

25. If the bicyclist starts from the starting-post and the runner starts from the half-way post, and they go opposite ways, where will they meet the second time?

26. One boy can run 5 yards while another runs 7; if both start opposite ways, from the same corner, to run around a block 75 yards on one street and 42 yards on the other, how far from the furthest corner will they meet?

27. The hands of a clock are together ten different times between midnight and noon. Find when.

28. The hands of a clock are at right angles twenty-two different times between midnight and noon. Find when.

29. The hands of a clock point opposite ways eleven different times between midnight and noon. Find when.

30. The hands of a clock make with each other an angle of 30° twenty-two different times between midnight and noon. Find when.

31. Find when on an actual clock the hands are exactly one minute-space apart. At what times are the conditions of the last four problems realized on an actual clock?

CHAPTER II.

ABBREVIATION OF RULES.

32. The second important use of Algebra is for abbreviating rules in Arithmetic.

For example, the rule for finding the number of square inches in a circle, when its radius is known, is: "Square the radius, and multiply by 3.1416." By Algebra we abbreviate this rule to

$$\pi r^2,$$

where r is understood to mean the radius of the circle and π stands for the number 3.1416 (or $3\frac{1}{7}$ as it is sometimes written).* Such an abbreviation of a rule is called a **formula**.

Translating Formulæ into Rules.

33. Model A.—Express as a rule the formula $a = \sqrt{(h + b)(h - b)}$, which enables us to determine the length, a, of one leg of a right triangle when the hypotenuse, h, and the other leg, b, are given.

Here $h + b$ is the sum of the hypotenuse and given leg, and $h - b$ is their difference; so, to find the required leg,—

Multiply the sum of the given sides by their difference and find the square root of the product.

* The exact value of the famous number represented by π cannot be expressed in figures. To ten places of decimals its value is 3.1415926535. The value $3\frac{1}{7}$ is near enough for ordinary use, and 3.1416 is more accurate still, being correct to within $\frac{1}{100000}$.

EXERCISE XVIII.

Translate into rules the following formulæ:

1. For the circumference of a circle, when the length of the radius is given: $2\pi r$.

2. For the area of an equilateral triangle, when the length of one side is given: $\dfrac{a^2\sqrt{3}}{4}$.

3. For the volume of a circular pillar, when the radius and height are given: $\pi r^2 h$.

4. For the volume of a square pyramid, when the height and one side of the base are given: $\dfrac{a^2h}{3}$.

5. For the volume of a sphere, when the radius is given: $\dfrac{4\pi r^3}{3}$.

6. For the diagonal of a rectangle, when the length and breadth are given: $\sqrt{l^2 + b^2}$.

7. For the average diameter of a tree, when the girth is known: $\dfrac{c}{\pi}$.

8. For the diameter of a ball, when the volume of it is known: $\sqrt[3]{\dfrac{6v}{\pi}}$.

9. For the number of crossings made by a number of straight lines drawn at random: $\dfrac{n(n-1)}{2}$. (n stands for the number of lines.)

10. For the number of seconds required for a body to fall: $\sqrt{\dfrac{2s}{g}}$. (s stands for the distance in feet, and g stands for the number 32.2.)

34. In writing a formula, numbers that are given in the rule are written in figures, except when the numbers are of

very frequent use and very well known value. Such are the numbers $\pi = 3.1416$, $g = 32.2$, and a few others.

Translating Rules into Formulæ.

35. Model B.—Give the formula for the following rule: To find the number of gallons a pail will hold: Find the diameter of the bottom and of the top, and find the depth of the pail; then multiply the two diameters together, square each diameter, and add the three results; then multiply by 11 times the height and divide by 10,000.

Let $D =$ the diameter of the top of the pail;
$d =$ the diameter of the bottom of the pail;
$h =$ the height of the pail.

Multiply the two diameters, Dd; square each, D^2 and d^2; add the three results, $D^2 + d^2 + Dd$; multiply by 11 times the height and divide by 10,000, $\frac{11h}{10000}(D^2 + d^2 + Dd)$.

EXERCISE XIX.

Give the formulæ for the following rules:

1. To find the area of an ellipse: Multiply half the greatest length by half the greatest breadth and multiply the product by 3.1416.

2. To find the tonnage of a vessel multiply the length of the keel by the breadth of the main beam, and by the depth of the hold in feet, and divide by 95.

3. To find the volume of a coffee-pot, multiply the greatest and least radii together; add to this product the square of the greatest radius and the square of the least radius; multiply the result by one-third the height; and finally multiply by 3.1416.

4. To gauge a cask, take the difference between the head and bung diameters, multiply by .62, and add the

head diameter; square this, multiply by the length of the cask and by $\frac{1}{4}$ of 3.1416.

5. To find the area of any triangle when the three sides are given: Add the three sides and divide by 2; subtract from this each side successively; multiply the four results together and find the square root of the product.*

6. To find the diameter of a dome when its surface is given, divide double the surface by 3.1416 and find the square root of the quotient.

7. To find the diameter of a cylinder when its height and volume are given, multiply the height by 3.1416 and divide four times the volume by the product; then find the square root of this result.

8. To find the depth of a conical hole when its cubical contents and its diameter are given, multiply the square of the diameter by 3.1416 and divide 12 times the volume by the product.

9. To find the velocity in feet per second of water flowing from a pipe, when the discharge is given in gallons per minute, and the diameter is given in inches, multiply the square of the diameter by 2.448 and divide the discharge by the product.

10. To find the velocity of water pouring from a hole in the side of a tank at a given depth (in feet) below the surface, multiply twice the depth by the number 32.2 and find the square root of the product.

Application of Rules.

36. Model C.—How many square feet of flooring are required to cover a circular platform 100 feet in diameter?

Here $r = 50$, $r^2 = 2500$, and
$$\pi r^2 = 2500 \times 3.1416 = 7854 \text{ square feet. } Ans.$$

* Let s stand for one-half of the sum of the three sides.

EXERCISE XX.

1. A large leaden ball was dropped into a tank completely full of water, and 33510.4 cubic centimetres of water overflowed. What was the diameter of the ball?

2. A circular floor contains 1256.64 square feet. What is its diameter?

3. It takes 1413.72 square yards of roofing-tin to cover a hemispherical dome. What is the diameter of the dome?

4. A cylinder 9 feet high and containing 763.4088 cubic yards is how wide?

5. A conical hole in the ground is intended to contain 261.8 cubic feet of water, and it is 10 feet across. How deep must it be in the centre?

6. Out of a lot of logs 12 feet long I want to pick out those not having less than 37.6992 cubic feet of lumber. What must be the least possible distance through?

7. I find that the average circumference of a lot of logs is 9.4248 feet. Suppose I cut the logs up into chopping-blocks, how wide would they be?

8. A horse attached to the arm of a windlass walks 10.56 feet short of 5 miles in making 300 turns. How long is the arm of the windlass?

9. A pipe 5 inches in diameter is to deliver 102 gallons of water per minute. What must the velocity of the water be, in feet per second?

10. How fast will water flow through a hole in a dam $32\frac{6}{7}$ feet below the surface of the water in the reservoir?

11. A milk-pail a foot deep is a foot across the top and 10 inches at the bottom. How many quarts will it hold?

12. A cask 30 inches long is 18 inches across at the head and 20 inches through at the bung. How many gallons will it hold?

13. How many gallons per 100 feet of length can be contained in a water-main 20 inches in diameter?

Application of Formulæ

37. Model D.—State the operations involved in the following formula and calculate its value for the given values of the letters:

$$W\frac{L^2+s^2}{12}; \qquad L=150; \qquad s=8; \qquad W=380.$$

Statement of Operation.—Multiply the number L by itself; multiply the number s by itself; add the two results, divide by 12, and multiply by the number W.

Evaluation.

$$\begin{array}{r} L^2 = 22500 \\ s^2 = \quad\ \ 64 \\ \hline 12)\underline{22564} \\ 1880\frac{1}{3} \\ W = \quad\ \ 380 \\ \hline 714526\frac{2}{3} \end{array}$$

EXERCISE XXI.

State the operations involved in the following formulæ, and calculate their values for the given values of the letters:

1. $W\frac{L^2}{12}$ $\qquad W=300; \quad L=20.$
2. Wr^2 $\qquad W=210; \quad r=5.$
3. $W\frac{L^2+s^2}{12}$ $\qquad W=200; \quad L=25; \quad s=2.$
4. $W\frac{r^2}{2}$ $\qquad W=350; \quad r=2.$
5. $W\frac{R^2+r^2}{2}$ $\qquad W=400; \quad r=20; \quad R=25.$
6. $W\frac{2r^2}{5}$ $\qquad W=80; \quad r=15.$

7. $W\left(L^2 + \frac{r^2}{2}\right)$ $W = 20$; $L = 10$; $r = 2$.

8. $IQ\frac{T - t}{T}$ $T=459$; $t=423$; $Q=161100$; $I=415$.

9. $\frac{V + nv}{V}$ $V = 2000$; $n = 300$; $v = 400$.

10. $C\frac{BD^2}{L}$ $C = 100$; $B = 3$; $D = 10$; $L = 15$.

11. $\frac{4L^2n^2d}{g}$ $L = 1500$; $n=1700$; $d=7.63$; $g=981$.

12. $\frac{2pP}{p + P}$ $p = 10$; $P = 12$.

13. $\frac{mr}{m - 1}$ $m = 1.5$; $r = 20$.

14. $\frac{pP}{P - p}$ $p = 8$; $P = 12$.

15. $\frac{d}{1 + n}$ $n = .02$; $d = 17$.

16. $\frac{Rr}{(R+r)(m-1)}$ $m = 2$; $r = 14$; $R = 84$.

17. $2dnm$ $d = 17326$; $n = 1440$; $m = 12.6$.

18. $n(4L + d)$ $d = 5$; $L = 30$; $n = 255$.

19. $f - \frac{d}{87} \cdot \frac{h}{30}$ $f = .372$; $d = 14.4$; $h = 29$.

20. $f - \frac{d}{96} \cdot \frac{h}{30}$ $f = .386$; $d = 15$; $h = 24$.

21. $\frac{360L}{c}$ $L = 40$; $c = 900$.

22. $\frac{\pi a}{180}$ $\pi = 3.1416$; $a = 60$.

23. $\frac{180u}{\pi}$ $u = 2.2$; $\pi = 3\frac{1}{7}$.

24. $\frac{180L}{\pi a}$ $L = 6$; $a = 10$; $\pi = 3.1416$.

25. $X\left(\frac{y+Y}{2}\right)$ $\quad y = 10; \quad Y = 14; \quad X = 7.$

26. $\frac{\pi a r^2}{360}$ $\quad \pi = 3.1416; \quad a = 12; \quad r = 10.$

27. $\frac{ca}{360}$ $\quad a = 8; \quad c = 100.$

28. $\frac{a^2 + c^2 - \frac{1}{2}b^2}{2}$ $\quad a = 4; \quad b = 13; \quad c = 15.$

29. $\frac{2a^2 + 2c^2 - b^2}{4}$ $\quad a = 10; \quad b = 17; \quad c = 21.$

30. $\frac{K^2}{h}$ $\quad h = 2; \quad K = 6.$

31. $\pi(R + r)(R - r)$ $\quad R = 25; \quad r = 20.$

32. $\frac{1}{2}h(L + l)$ $\quad b = 3.5; \quad l = 35.25; \quad L = 24.75.$

33. $4\pi^2 r(R + r)$ $\quad 2r = 3; \quad 2R = 8.$

34. $\pi h(R + r)(R - r)$ $\quad h = 10; \quad R = 5; \quad r = 3.5.$

35. $\frac{h}{6}(B_1 + B_2 + 4M)$* $\quad \begin{cases} B_1 = 56; \quad B_2 = 30; \\ M = 42; \quad h = 9. \end{cases}$

(Read "B one," "B two"; or "B sub one," "B sub two.")

36. $\frac{1}{3}\pi r^2 h$ $\quad \pi = 3.1416; \quad r = 10; \quad h = 8.$

37. $2\pi rh$ $\quad \pi = 3.1416; \quad r = 8; \quad h = 7.$

38. $r\frac{v + w}{V - w}$ $\quad \begin{cases} r = 10; \quad v = 8; \quad w = 13; \\ V = 21. \end{cases}$

39. $\frac{Q}{4r^2\pi}$ $\quad Q = 110; \quad r = 5; \quad \pi = 3.1416.$

40. $\pi\left(ab + \frac{b^2}{4}\right)$ $\quad a = 18; \quad b = 15.$

41. $\frac{RA}{a - A} - r$ $\quad \begin{cases} R = 120; \quad A = .125; \quad a = .875; \\ r = 15. \end{cases}$

42. $R\frac{A}{a - A} - (R' + r)$ $\quad \begin{cases} a = .287; \quad A = .234; \quad R = 100; \\ r = 100; \quad R' = 110. \end{cases}$

(Read "R prime"; or "R large prime.")

* The same letter can be used to denote different numbers even in the same formula by using distinguishing marks, such as suffixes, as in 35, or accents, as in 42.

43. $\dfrac{cR'}{R + R'}$ $\quad c = 3; \quad R' = 100; \quad R = 21.$

44. $\dfrac{Rr + R'r + R'R}{RR'r}$ $\quad R = 100; \quad r = 120; \quad R' = 1000.$

45. $\dfrac{E}{\dfrac{R}{x} + \dfrac{r}{y}}$ $\quad E = 1.8; \ x = 2; \ y = 5; \ R = 10; \ r = 100.$

46. $\dfrac{1}{2}\left(\dfrac{a}{n - a} + \dfrac{b}{n - b}\right)$ $\quad a = .002; \quad b = .015; \quad n = 60.$

47. $\dfrac{1}{t}\left(\dfrac{T^2}{R^2} - 1\right)$ $\quad T = 8; \quad R = .75; \quad t = 10.$

48. $\dfrac{x(T - t)}{W(Q - T)}$ $\quad \begin{cases} x = 243; \ T = 16; \ t = 12; \\ Q = 96; \ W = 125. \end{cases}$

49. $\dfrac{H - h\dfrac{d}{D}}{1 + tc}$ $\quad \begin{cases} c = .002; \ t = 20; \ H = 60; \\ h = 3; \ d = 1.5; \ D = 13.5. \end{cases}$

50. $\dfrac{(Q - t)(w + W)}{V\left(T - \dfrac{Q + t}{2}\right)}$ $\quad \begin{cases} T = 98; \quad t = 10; \quad Q = 26; \\ w = 14; \ W = 500; \ V = 126. \end{cases}$

51. $S\dfrac{W(T - Q)}{w(Q - t)} - \dfrac{V}{w}$ $\quad \begin{cases} V = 20; \ w = 300; \ W = 10; \\ S = 19; \ t = 10; \ Q = 13.8; \ T = 20. \end{cases}$

52. $\dfrac{W}{w}(T - q) - s(r - t) - S(q - r)$

$$\begin{cases} s = 5; \quad S = 1; \quad W = 500; \quad w = 240; \quad t = 0; \\ T = 44; \quad q = 6; \quad r = 5.5. \end{cases}$$

53. $\dfrac{W}{w}(q - t) - s\left(T - \dfrac{t + q}{2}\right)$

$$\begin{cases} W = 1000; \quad w = 25.6; \quad T = 100; \quad q = 20; \quad t = 4; \\ s = 5. \end{cases}$$

54. $\dfrac{M}{D^3\left(1 + \dfrac{d^2}{D^2}\right)}$ $\quad D = 10; \quad d = 1.5; \quad M = 2.$

55. $l[1 + c(T - t)]$ $\quad \begin{cases} l = 48.7; \quad c = .00002; \\ T = 15; \quad t = 10. \end{cases}$

56. $\dfrac{P}{W(t_2 - 2t_1 + t)}$ $\begin{cases} P = 1244; \quad W = 2; \quad t_2 = 14.5; \\ t_1 = 11.5; \quad t = 10. \end{cases}$

57. $\dfrac{d(W' - w)}{W - w}$ $\begin{cases} w = 32; \quad W = 132; \quad W' = 1390; \\ d = .995. \end{cases}$

58. $\left(\dfrac{V}{V + v}\right)^n$ $v = 500; \quad V = 2000; \quad n = 5.$

59. $\dfrac{abc}{4K}$ $a=15; \quad b=28; \quad c=41; \quad K=126.$

60. $\dfrac{K}{s - a}$ $\begin{cases} s = \dfrac{a + b + c}{2}; \quad a = 19; \\ b = 20; \quad c = 37; \quad K = 114. \end{cases}$

Formula for the Area of a Triangle.

38. An extremely important formula is that for finding the area of a triangle when the lengths of the three sides are given : the area of the triangle being represented by K and the lengths of the sides by a, b, and c, we have the formula

$$K = \sqrt{s(s - a)(s - b)(s - c)},$$

where s represents

$$\frac{a + b + c}{2}.$$

EXERCISE XXII.

By this formula find the area of the triangles whose sides are :

1. $a = 12; \quad b = 17; \quad c = 25.$
2. $a = 25; \quad b = 51; \quad c = 74.$
3. $a = 6\ ; \quad b = 25; \quad c = 29.$
4. $a = 16; \quad b = 25; \quad c = 39.$
5. $a = 17; \quad b = 25; \quad c = 28.$

Formula for Square Root.

39. The rule for square root can be conveniently memorized by means of a formula:

$$
\begin{array}{l|l}
5'\,61'\,69 & 2 \\
4 & \\
\hline
161 &
\end{array}
$$

Rule. I. Separate the number into periods of two figures each, beginning at the decimal point. Begin with the left-hand period, subtract its square, and bring down the next period.

II. Let a stand for 10 times the part of the root already found. Write $2a$ for a new divisor, and call the quotient b; then subtract $b(2a + b)$ from the new dividend.

$$
\begin{array}{rl|l}
 & 5'\,61'\,69 & 23 = a + b \\
a^2 = & 4\ 00 & \\
\cline{2-3}
 & 1\ 61 & 40 = 2a \\
b(2a + b) = & 1\ 29 & 43 = 2a + b \\
\cline{2-3}
 & \ \ 32 &
\end{array}
$$

III. Bring down the next period for a new dividend. Let a stand for 10 times the part of the root already found, and write $2a$ for a new divisor; call the quotient b, and subtract $b(2a + b)$ from the new dividend.

$$
\begin{array}{rl|l}
 & 5'\,61'\,69 & 237 = a + b \\
\left[\begin{array}{l} 40000 + 12900 = \\ 52900 = 230^2 \end{array}\right] & 4\ 00\ 00 & \\
\cline{2-3}
 & 1\ 61 & 40 \\
 & 1\ 29\ 00 & 43 \\
\cline{2-3}
 & 3269 & 460 = 2a \\
 & & \quad 7 = b \\
b(2a + b) = & 3269 & 467 = 2a + b \\
\cline{2-3}
 & \quad 0 &
\end{array}
$$

40. This process can be shortened by leaving off the cipher on $2a$, or writing it lightly, so that when b is added

it can be put in place of the cipher; then the completed work would read:

Model E.

$$\begin{array}{l|l}
5'\,61'\,69 & 237 \\
4 & \\
\hline
1\;61 & 43 \;\;= 2a + b \\
1\;29 & \;\;3 \;\;= b \\
\hline
3269 & 467 = 2a + b \\
3269 & \;\;\;\;7 = b \\
\hline
0 &
\end{array}$$

Of course the pupil must remember that at each new stage of the process a new meaning is given to the letters a and b.

EXERCISE XXIII.

Find the square roots of the following numbers:

1. 6889.	6. 546121.	11. 79.21	16. 811801.
2. 44944.	7. 958441.	12. .049729	17. 3684.49
3. 138384.	8. 982081.	13. 11.9716	18. 1.062961
4. 245025.	9. 1948816.	14. .00710649	19. 418.6116
5. 375769.	10. 1292769.	15. 9545.29	20. 19796.49

Approximate these square roots to three decimal places:

21. 2.	23. 5.	25. 10.	27. 1.032	29. 7.921
22. 3.	24. 6.	26. 9.87	28. 22.5	30. 688.9

EXERCISE XXIV.

Evaluate the following formulæ:

1. $\sqrt{s(s-a)(s-b)(s-c)}$ $\quad a = 29;\ b = 60;\ c = 85.$

2. $\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$ $\quad a = 35;\ b = 44;\ c = 75.$

3. $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ $a = 48;\ b = 85;\ c = 91.$

4. $\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$ $a = 40;\ b = 111;\ c = 145.$

5. $\sqrt{rr_1r_2r_3}$ $\begin{cases} r = 12;\ r_1 = 18;\ r_2 = 46\frac{2}{3}; \\ r_3 = 157\frac{1}{2}. \end{cases}$

6. $\sqrt{r^2 - \frac{1}{4}k^2}$ $k = 240;\ r = 125.$

7. $\sqrt{h^2 + \frac{1}{4}k^2}$ $k = 130;\ h = 7.$

8. $\frac{2kr}{\sqrt{4r^2 - k^2}}$ $r = 40;\ k = 9.$

9. $\sqrt{\frac{6Pg}{W}}$ $\begin{cases} g = 981;\ P = 1308; \\ W = .0002. \end{cases}$

10. $\frac{1}{2l}\sqrt{\frac{a^3b}{\pi r^2k}}$ $\begin{cases} l = 25;\ a = 10;\ b = .981; \\ \pi = 3.1416;\ r = 5;\ k = 8. \end{cases}$

11. $d\sqrt{\frac{8\pi f}{A}}$ $d = 1.5;\ A = 220;\ f = .15.$

12. $\frac{\pi(a+b)ab}{\sqrt{a^2+b^2}}$ $a = 15;\ b = 37;\ \pi = 3.1416.$

13. $\sqrt{\frac{w}{\pi ld}}$ $\begin{cases} w = .231;\ \pi = 3.1416; \\ l = 4;\ d = 13.6. \end{cases}$

14. $\sqrt{2r^2 - r\sqrt{4r^2 - k^2}}$ $r = 100;\ k = 160.$

15. $\sqrt{\frac{P}{D}(1.41)\frac{a+t}{t}}$ $\begin{cases} a = 273;\ t = 27;\ D = .00188; \\ P = 1{,}045{,}500. \end{cases}$

41. Model F.—In the following formula what value must be given to n when the other letters have the values $T = 210$, $t = 85$, $c = .0002$ in order that T' may have the value 210.75 ?

$$T' = (T - n) + n\{1 + c(T - t)\}.$$

Let the required value of n be represented by x; then

$$(210 - x) + x\{1 + .0002(210 - 85)\} = 210.75$$
$$210 - x + x\{1.025\} = 210.75$$
$$210 + .025x = 210.75$$
$$.025x = .75$$
$$25x = 750$$
$$x = 30$$

EXERCISE XXV.

1. $\mathrm{G} = \dfrac{\mathrm{h}^2(\mathrm{l} + \mathrm{k}) + \frac{1}{4}\mathrm{k}^2(\mathrm{l} - \mathrm{k})}{4\mathrm{h}}$

$G = 86.3895$; $h = 2$; $k = 20$. Find l.

2. $\mathrm{G} = \frac{1}{6}\pi \mathrm{h}[3(\mathrm{r}_1{}^2 + \mathrm{r}_2{}^2) + \mathrm{h}^2]$

$r_1 = 40$; $r_2 = 30$; $h = 30$; G is found to be 131,880. What value was used for π?

3. $\dfrac{\mathrm{a}}{4}(\mathrm{B} + 3\mathrm{T}) = \mathrm{V}$

When $a = 340$, $B = 12{,}025$, and $V = 1{,}615{,}000$, what must be the value of T?

4. $\mathrm{D} = \dfrac{\mathrm{V} + \mathrm{nv}}{\mathrm{V}}$

When V is 3000 and $v = 15$, what value of n will make $D = 10$?

5. $\pi\left(\mathrm{ab} + \dfrac{\mathrm{b}^2}{4}\right) = \mathrm{S}$

What value must be given to a in order that S may be 7854 when b is 50? ($\pi = 3.1416$).

The Interest Formula.

42. In the formula $prt = i$, p stands for the number of dollars in the principal, t for the number of years, r for the rate per centum expressed as a decimal, i for the interest.

Model G.—Find the interest on \$300 for 1 year and 3 months at 5 per cent.

$$\begin{aligned} p &= 300;\ r = .05;\ t = 1.25 \\ i &= 300(.05)(1.25) \\ &= 18.75 \end{aligned}$$

Ans. \$18.75.

43. The formula for the amount is $a = p + prt$, which can also be written $p(1 + rt) = a$.

For the amount at compound interest the formula is

$$a = p(1 + r)^t.$$

EXERCISE XXVI.

1. What is the interest on \$532 for 3 years 1 month and 15 days at 5 per cent?

2. What is the amount of \$298 at $3\frac{1}{2}$ per cent for 6 months?

3. What sum of money will return \$75 interest at 2 per cent in 2 years 6 months?

4. If \$1015 is put at interest at 7 per cent, how long before it earns \$100 interest?

5. If \$784 earns \$15.68 interest in 3 years, what is the rate exacted?

6. What sum of money, if put at interest at 3 per cent for 5 years, will amount to \$1000?

7. What rate of interest must be charged on \$1500 for 5 years to make it amount to \$2047.50?

8. How long before the principal, p, will double itself at 5 per cent simple interest?

9. How long will it take \$1500 to amount to \$1591.35 at 3 per cent compound interest?

10. Show that the formula for r in compound interest is

$$r = \sqrt[t]{\frac{a}{p}} - 1.$$

CHAPTER III.

TRANSFORMATIONS.

44. It is often necessary to change the form of a formula or other algebraic expression, or to perform some algebraic operation upon it. In order to be able to do such things intelligently we must investigate the laws and the methods of addition, subtraction, multiplication, and division, in order to see clearly how they differ from the similar laws and methods of ordinary arithmetic.

Negative Quantities.

45. The expression $a - b$ is the algebraic symbol for the result of subtracting b from a. But if b is greater than a the subtraction is impossible. The question is in that case what does $a - b$ represent?

46. Take a particular case. Suppose I send an order for 10 quarts of berries. If the grocer has a quarts in stock he will have after filling my order $a - 10$ quarts.

Suppose his original stock was only 7 quarts. He would send me those, but his condition would not be as if he never had had any berries, nor any orders for berries; for he now has an unfilled order for 3 quarts.

To put these circumstances into arithmetical shape :

47. It is required to subtract 10 from 7. The operation is impossible. But 7 can be taken from 7, leaving 0,

and there are still 3 to be taken FROM ANY NUMBER THAT MAY HEREAFTER BE ADDED TO THE EXPRESSION; just as the grocer is expected to fill his order from any goods that may hereafter come in.

$$\text{(1)}\quad 7 - 10 = -3$$

48. The expression -3 is new to the student. It means 3 to be subtracted from any number that may hereafter be added to the expression.

If we add 5 to the expression in (1), we have

$$\text{(2)}\quad 7 - 10 + 5 = -3 + 5$$
$$\text{(3)}\quad -3 + 5 = 2$$

If we add 3 to the expression in (1), we have

$$\text{(4)}\quad -3 + 3 = 0$$

49. The expression -3 differs from all expressions known to arithmetic in this particular, namely, that you can get zero by ADDING to it.

50. When two quantities added together give zero, one is called the **negative** of the other; but generally by a negative quantity we mean a quantity preceded by the minus sign.

That is, 3 is the negative of -3 just as much as -3 is the negative of 3; but if asked which of the two was the negative we should say -3.

51. Terms preceded by minus signs are also called **minus** or **negative** terms; and terms not preceded by minus signs are called **plus**, or **positive** terms.

52. Thus it may be said that any POSITIVE term is the NEGATIVE of the corresponding minus term; the apparent contradiction being explained when we remember the definition.

53. Since it is immaterial in what order two numbers are added, we may as well add -3 to 3 as 3 to -3; this would lead to the expression $3 + (-3) = 0$, where the new kind of quantity is added.

54. Finally, we may translate the sign $-$ by the words "the negative of."

EXERCISE XXVII.

1. If $a + b = 0$ what is the negative of a?
2. Find the value of $c + x$ when $c = 10$; $x = a - b$; $a = 3$; $b = 5$.
3. What is the arithmetical meaning of -11?
4. What is the negative of -11?
5. Translate the equation $-(-4) = 4$.
6. Simplify $-(-a)$; $-(-b)$.
7. Simplify $-(-a - b)$.
8. $-(x - a)$; simplest form.
9. Find the value of $x + y + a$ when $x = -5$; $y = 2$; $a = c$.
10. If $x + y - z = 0$, what is the negative of x?

Reciprocals.

55. When the product of two numbers is 1, either number is called the **reciprocal** of the other.

Thus if $ab = 1$, a is the reciprocal of b.

The reciprocal of 7 is $\frac{1}{7}$; that of $2\frac{1}{3}$ is $\frac{3}{7}$; that of x is $\frac{1}{x}$; that of .25 is 4; that of 1.125 is $\frac{8}{9}$; and so on.

56. The reciprocal of the reciprocal of a number is the number itself.

EXERCISE XXVIII.

1. If $ab = 1$, what is the reciprocal of a?
2. If $5x = 1$, what is the reciprocal of x?
3. What is the reciprocal of 2.5?

4. $\frac{1}{5} = a$; what is the reciprocal of a?

5. What number added to its reciprocal gives 10.1?

Rearrangements.

57. In the following "chain of additions and subtractions" there are ten terms, some positive and some negative:

$$5 + 17 - 3 + 2 - 30 + 15 - 3 + 5 + 2 - 1$$

In such expressions

I. The terms may be **rearranged in any order,** without changing the value of the entire expression.

II. The terms may be **grouped in any way,** the values of the groups found, and all the results added together, giving the same result as the original expression.

That is, $5 - 3 + 5 - 30 - 1 + 2 - 3 + 15 + 2 + 17 = 9$ and $(5 - 30) + (17 - 3 + 2 + 15) + (-3 + 2 - 1) + 5$ or $-25 + 31 + (-2) + 5 = 9$.

EXERCISE XXIX.

Arrange each of the following expressions in six different ways without changing the value of the expression:

1. $x + 5y + z$.
2. $x + y - z$.
3. $p - q + s$.
4. $10h - 20q + 21r$.
5. $z + 2a - 2b$.

Make a group of two terms in each of the following expressions without altering the value of the expression; six answers for each:

6. $a + b + c + d$.
7. $h - k - j + x$.
8. $3x + 2y - c - 5d$.
9. $13p - 31q + 14r - 41s$.
10. $12x + 23y - 31z + 4w$.

58. The same truths hold of a chain of multiplications and divisions, and may be restated as follows, understanding by the word "term" one of the numbers together with the $\times$ or $\div$ sign preceding it:

I. The terms in a chain of MULTIPLICATIONS and DIVISIONS may be **rearranged in any order**, so long as each divisor remains a divisor and each multiplier a multiplier.

II. The terms in a chain of multiplications and divisions may be **grouped in any way** and the groups multiplied, so long as each divisor remains a divisor and each multiplier a multiplier.

$$1 \div 3 \times 15 \times 2 \div 5 \times 7 \div 14 \times 3 \div 7 \times 21 \div 2 \times 6 \div 9$$

may be written

$$1 \div 5 \div 14 \div 7 \times 21 \times 7 \div 3 \times 15 \times 2 \times 3 \div 2 \times 6 \div 9$$

or $(1 \div 3 \times 15 \times 2) \times (\div 5 \times 7 \div 14 \times 3 \div 7) \times (21 \div 2 \times 6 \div 9)$ or $10 \times \frac{3}{70} \times 7$ which equals 3; understanding that $\div 5 \times 7$ is the same as $7 \div 5$.

EXERCISE XXX.

Arrange the following products each in six different ways:

1. abc.
2. $5ab$.
3. $17xy$.
4. $5x(a - b)$.
5. $3(x - y)(a - b)$.
6. $x(a - b)(x - z)$.
7. $(a - b)(b - c)(c - a)$.

The purely numerical multiplier, or the coefficient usually so called, is generally put first; with this restriction rearrange in six ways (including the order given) the following:

8. $5xy(a - c)$. 9. $21st(u - v)$. 10. $301h(k - j)(j - k)$.

Show why the last two multipliers in 10 *are not alike.*

SUMMATION.

59. The introduction of negative quantities makes addition and subtraction in algebra more difficult than in arithmetic; and the necessity of having terms similar before they can be united introduces a further complication.

60. The actual operation of addition in arithmetic consists in taking two numbers that have been counted up separately, putting them together, and counting them up as one number. For example, two distances are measured from the beginning of the first to the end of the first, and from the beginning of the second to the end of the second; when they are added the beginning of the second is put on the end of the first and the united distances are measured from the beginning of the first to the end of the second.

61. When expressions are added in algebra they are written one after another, each term retaining its proper sign, and the similar terms are then united as if all the expressions were one.

Model A.—Add the following expressions:

$3x + 5 - 2x + 1;\ -2 + 2x - 10 + 5x;\ x + 11 - 3x + 1.$

Operation.—

① $3x + 5 - 2x + 1 - 2 + 2x - 10 + 5x + x + 11 - 3x + 1$

② $3x - 2x + 2x + 5x + x - 3x + 5 + 1 - 2 - 10 + 11 + 1$

③ $6x + 6$

The form ② came by rearranging ① according to I, § 58, so that all the x-terms would be together; then ③ came by uniting the two groups of terms in ② according to § 21. The result cannot be further simplified till we know the value of the letter x.

EXERCISE XXXI.

Add the following expressions, and indicate the three steps of the process in each example:

1. $3x + 2 + 4x + 7;\ x + 3;\ 3x - 12;\ 1 - 10x.$
2. $5x + 7x + 13x - 5x;\ 3 - 17x + 2 - 3x;\ x - 5;\ -x.$

3. $8+2x+3+9x-7-x-4-3x$; $x+5-8x$; $2x+1$; $-x$; $-x$.
4. $3+5x+13x+17-8x+5-10x$; $4x+2-3x+7$; $34-x$.
5. $171x-243+318x-411x-111+150$; $38x+117$; $301-100x$; $102x-299$.
6. $3(x-5)$; $5(x-5)$; $8(5-x)$.
7. $17(2x-3)$; $11(4-3x)$; $(x-1)3$.
8. $7a+341+2a-100-200a-200$; $10(a-3)$; $3(3-a)$.
9. $3y+7$; $(7+y)3$; $7(3+y)$; $-(-3)$; $-3-7-y$.
10. $-(-3s)$; $-(-8)$; $-(-3s-8)$; $3(s+8)$.

62. Only similar terms can be united.

EXERCISE XXXII.

Add the following expressions:

1. $3x+8$; $3y+8$; $2x+4y$; $x-y-15$; $-5x-3y+3$.
2. $2x+y-z$; $x-3y+z$; $4y-3x+z$.
3. $13x-4y+z$; $4y-10x-5z$; $4z-4x+4y-4$.
4. $-131x-100y-z$; $-(100z-100x)$; $50x+150y$.
5. $x-2y$; $2y+x$; $-x$; $2x-y$.
6. $2p-3q$; $p-q$; $5p-2q$; $p+q+r+s$.
7. $3x+13y-(-17)$; $20-10x-10y$; $x-y-1$; $-(-2)$.
8. $10x+r+z$; $x+10r+c$; $x+r+10z$; $x+r+z+13$.
9. $3p+33r+333s$; $2r+22s+222p$; $s+11p+111r$; $-100r-100s-100p$.
10. $-139h-128k$; $k-g$; $g+100h$; $120k-h-2g$; $5g+5h+5k+5$.

SUBTRACTION.

63. In subtraction the **minuend** is the quantity from which the **subtrahend** is taken to leave the **remainder.**

When any quantity is added to the subtrahend, the effect on the remainder is precisely the same as if the same quantity was taken from the minuend; and when any quantity is taken from the subtrahend, the effect on the remainder is precisely the same as if the quantity was added to the minuend. For instance, the following subtractions:

Minuend.	50	$50-2$	50	$50+2$
Subtrahend.	$30+2$	30	$30-2$	30
Remainder.	$20-2$	$20-2$	$20+2$	$20+2$

64. It is possible, therefore, without changing the value of the remainder, to take any term out of the subtrahend and write it as a part of the minuend; only we must remember to change its sign, from $+$ to $-$, or from $-$ to $+$. And if we can so transfer any term, we may transfer them all, and thus convert our example in subtraction into an example in addition.

Model B.

Minuend. $7x-3xy+5\ -5a-10x^2$

Subtrahend. $3-8xy+2x+5a-\ 2x^2$

Remainder. (To be obtained.)

may be converted into

Minuend. $7x-3xy+5-5a-10x^2-3+8xy-2x-5a+2x^2$

Subtrahend. (All these terms have gone into the minuend.)

Remainder. (New minuend simplified.) $5x+5xy+2-10a-8x^2$

65. The rule for subtraction in Algebra is therefore given briefly as follows:

Change signs in the subtrahend and unite similar terms.

66. To save time in calculation the signs of the subtrahend should be changed IN YOUR MIND, and the terms united without writing the expressions over again.

EXERCISE XXXIII.

Perform the following subtractions:

1. From $x^3 - 7x^2 + 16x - 12$ take $x^3 - 9x^2 + 2x - 48$.
2. From $a - b$ take $a - b - c$.
3. From $x - y$ take y.
4. From h take $- s$.
5. From $x^3 + 9x^2 + 2x - 48$ take $x^3 - 4x^2 - 8x + 8$.
6. Take $x^3 - 5x^2 - 2x + 24$ from $x^3 + 2x^2 + 4x + 3$.
7. Subtract $24 + 14x - 29x^2 + 6x^3$ from $2x^3 + 3x^2 - 13x - 12$.
8. $2 - \sqrt{3}$ from $6 + 5\sqrt{3}$.
9. Take a from $a - b$.
10. From $9p - 5q + 4r$ take $5q - 2p + 2r$.

Parentheses.

67. To save using words like *subtract, take*, etc., the operation of subtraction is often indicated by putting the subtrahend in a **parenthesis** with the sign $-$ before it. In such examples the parenthesis, WITH ITS SIGN, serves only to mark the subtrahend and distinguish it from the minuend.

EXERCISE XXXIV.

1. $2a - 2b + 3c - d - (5a - 3b + 4c - 7d)$.
2. $x^4 + 4x^3 - 2x^2 + 7x - 1 - (x^4 + 2x^3 - 2x^2 + 6x - 1)$.
3. $-(a^2 - ax + x^2) + 3a^2 - 2ax + x^2$.
4. $4x^3 - 2x^2 + x + 1 - (3x^3 - x^2 - x - 7) - (x^3 - 4x^2 + 2x + 8)$.
5. $10a^2b + 8ab^2 - 8a^3b^3 - b^4 - (5a^2b - 6ab^2 - 7a^3b^3)$.
6. $6x^2y - 3xy^2 + 7y^3 + x^3 - (8x^2y - 3xy^2 + 9y^3 + 11x^3)$.
7. $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2 - (- \frac{3}{2}x^2 + xy - y^2)$.
8. $\frac{2}{3}a^2 - \frac{5}{2}a - 1 - (- \frac{2}{3}a^2 + a - \frac{1}{2})$.
9. $\frac{1}{3}x^2 - \frac{1}{2}x + \frac{1}{6} - (\frac{1}{3}x - 1 + x)$.
10. $\frac{3}{8}x^2 - \frac{2}{3}ax - (\frac{1}{3} - \frac{1}{4}x^2 - \frac{5}{6}ax)$.

68. Expressions often occur in Algebra where it is desirable to take out parentheses from them and simplify by uniting terms. To do so it is necessary to remember first that a parenthesis with $-$ before it indicates SUBTRACTION, and the signs of all the terms within must be changed when THE PARENTHESIS, WITH ITS SIGN, is removed; and, secondly, that a parenthesis with $+$ before it indicates ADDITION, and no change of sign is necessary when THE PARENTHESIS, WITH ITS SIGN, is removed.

69. When parentheses occur within parentheses, it is best, to avoid confusion, to begin within the inner one, that is, not to remove a parenthesis which has another parenthesis within it.

70. The sign of the first term in a parenthesis is NOT the sign of the parenthesis; thus in the two expressions

$$3z - \overline{x - y};\;\; 5z - \{x + 5\}$$

the sign of x is $+$.

EXERCISE XXXV.

Simplify the following expressions:

1. $x + 1 + (5 - x) - (3 - 13x)$.
2. $x + y - 5 - (x - y + 5)$.
3. $2a - 3b + (3a - 2b) - (2a - 3b)$.
4. $a - b + c - (a + b - c)$.
5. $14a + 27b - 13 - (7a - 110b - 17)$.
6. $20x + 30y - 15z - (15y + 16z - 17x)$.
7. $1108 + 135x - 780y - (39x + 42y - 109)$.
8. $3x^2 + 33x + 333 - (333x^2 - 33x + 3)$.
9. $2700x^2 - 908y^2 - 137xy$
   $- (1900x + 3000x^2 + 100y + 1000y^2)$.
10. $x^2 + y^2 + 2x + 2y - (x^2 - y^2 - 2x + 2y)$.

* 11. $2x + 3y - 5 - 9(5 - y + x)$.

* See § 13.

12. $a - 5b + 3c + (a - b) - 2(5b - c - a)$.
13. $x - y - 5(3y - 17x)$.
14. $2x + 3x^2 + 1 + 2(x^2 + 1) - 5(1 + x^2)$.
15. $4(x - 1) - 5(2 - 3x)$.
16. $a - b - 3(2a - b) + 7(a + b)$.
17. $x + 2y + (x - a) - 2(a - 3y)$.
18. $x + 5 - 3(1 - 2x)$.
19. $x^2 + y^2 - 2(x + y)$.
20. $a + 5b + 3c - x - 2(x - a - 5b + 3c)$.

Nests of Parentheses.

71. Model C.

$$a - [b - \{c + (d - \overline{e - f})\}]$$
$$a - [b - \{c + (d - e + f)\}]$$
$$a - [b - \{c + d - e + f\}]$$
$$a - [b - c - d + e - f]$$
$$a - b + c + d - e + f$$

EXERCISE XXXVI.

Simplify the following expressions:

1. $x - (y - z) + x(y - z) + y - (z + x)$.
2. $7a - [2b + \{a - (b + a)\}]$.
3. $a - [3a - \{5b - (4c - 3a)\}]$.
4. $\{k - (s - t)\} + \{s - (t - k)\} - \{t - (k - s)\} - (k + s + t)$.
5. $2x - (6y + [4z - 2x]) - (6x - [y + 2z])$.
6. $-[-\{-(b + c - 2a)\}] + [-\{-(c + a - 4b)\}]$.
7. $-(-(-2a)) - (-(-(-3x)))$.
8. $4x - [2x + 2y - \{x + y + z - (2x + 2y + 2z + k)\}]$.
9. $-3a - [4x + \{4c - (5y + 4x + 3a)\}]$.
10. $-[6x - (12y - 4x)] - [6y - (4x - 7y)]$.

Where the answers to the following equations are not whole numbers, express them in decimal fractions:

11. $x + 3 - (2x - 17) = 4$.
12. $2x - 7 - (x + 2) = 2x - (2x + 9)$.

13. $3x + 2 - (2x - 5) + (9 - 7x) = x + (2 - 5x)$.
14. $13 - (5 - 2x) - (1 - 7x) = 25 - (2 + 7x)$.
15. $300 - 2x - (115 - 3x) = 1000 - (3x - 61)$.
16. $100x - (39 + 8x) - 10 = 12x + 345$.
17. $x - (3x - \overline{2x + 1}) = 8 - (3x + 7)$.
18. $12x - 3(x - 2) = 20 - (5 + \overline{9 - 7x})$.
19. $100 - (2 - \overline{15x - 23}) = 98 - (20x - 24 - 5x)$.
20. $25x - 10(1 + x) = 3 \quad (x - \overline{5x - 7}) + 8x$.

72. To put terms of any expression into brackets similar precautions are necessary. If a $+$ sign is used before the bracket, no change is necessary; but if a $-$ sign is used, that takes part of the MINUEND and makes it a SUBTRAHEND; consequently every $+$ must be changed to $-$, every $-$ to $+$. Of course there are various ways of bracketing the terms of any expression.

Model D.

$ax^2 - 2hxy + by^2 - 2gx - 2fy + c$ may be written

$$(ax^2 - 2hxy) + (by^2 - 2gx) + (-2fy + c)$$

or $$(ax^2 + by^2 + c) - (2hxy + 2gx + 2fy)$$

or $$(ax^2 - hxy - gx) - (hxy - by^2 + fy) - (gx + fy - c)$$

or in many other ways.

EXERCISE XXXVII.

In the following examples get two results, first with a $+$ sign before each bracket, then with a $-$ sign:

1. Bracket terms containing x:

$$bcx + abx + acy - abc.$$

2. Bracket terms containing a in the same expression.
3. Bracket terms containing ab in the same expression.

4. $a^3bc^3 - ab^3c^3 + a^3b^3c - a^3b^3c - a^3bc^3 + a^2b^2c^3 + a^3b^2c^2$; bracket terms containing a^3.

5. Bracket last two terms: $b^2c^2 - a^2b^2 - a^2c^2$.

6. Bracket last three terms: $x^2 - y^2 + 2yz - z^2$.

7. Bracket separately terms containing a and terms containing b: $ax + by - ay - bx + az - bz$.

Bracket like powers of x:

8. $ax^3 + bx^2 + c + bx^3 + cx^2 + a + cx^3 + ax^2 + b + ax + bx + cx$.

9. $ax^2 + 5x^3 - a^2x^4 - 2bx^3 - 3x^2 - bx^4$.

10. $ax^2 + a^2x^3 - bx^2 - 5x^2 - cx^3$.

73. The letters a, b, c, p, and q in the following expressions are called **constants,** and the letters x, y, and z are called **variables.**

EXERCISE XXXVIII.

Group the variable terms in one parenthesis, obtaining two results as before, one with a + bracket and the other with a − bracket.

1. $a^3 + b^3 + x^3 + y^3 - 3a^2bx - 2ab^2x + 5a^2b - 2x^2y$.
2. $a^2 + c^2 + b^2 + x^2 - 2ax - 2ab - 2bc + 2bc + 2bx + 2cx$.
3. $a^2x^2 + 2bx^3 - x^3 + 2a^2b + 4b^2 - 2bx - a^3 - 2ab + ax$.
4. $a(x + c) - p(a + y) - c(x - y + p - q)$.
5. $3(x - a) + 5(a - 2x - 3y) - 17(b + c - 2z)$.
6. $2a(x + b - 2c) - 2c(y + q - 2b)$.
7. $5x(x + y - a) + 7b(x - a + 2p - 5q) - 8pq$.
8. $7a(2x - p + 2q) - 3p(2y - a + 2bc) - 5a^2$.
9. $2(4a - 3x + 2p) - 5(2b - 5y + 3q) - 7a^2$.
10. $12x(a^2 - b^2) + 5a(a^2 - b^2) - 2a(a + b - 2px)$.

MULTIPLICATION.

74. The multiplier and the multiplicand are called the FACTORS of the product; in continued multiplication there are more than two factors.

E.g., $3 \times 5 = 15$; $15 \times 7 = 105$; $105 \times 11 = 1155$.

3 and 5 are the factors of 15; 15 and 7, or 3, 5, and 7, are the factors of 105; 3, 5, 7, and 11 are the factors of 1155.

It appears from the principles stated in § 58 that the factors of any product may be written in any order, so that, e.g., $5ab = 5ba$.

75. When more than one of the factors of a number are alike, it is sometimes convenient to indicate the fact as follows:

$$2 \times 3 \times 3 \times 3 = 2 \times 3^3 = 54;\ 2aaabbbb = 2a^3b^4.$$

76. The small figure denoting the number of equal factors is called an INDEX.

77. The product of equal factors is called a POWER.

78. One of the equal factors of a power is called a ROOT.

For instance, 27 is the third power of 3, 2 is the fifth root of 32; 64 is a sixth power, 2 being the root and 6 the index; 36 is a second power, 6 being the root and 2 the index.

79. The second power is generally called the square, and the second root the square root; 8 is the square root of 64.

The third power is often called the cube, and the third root the cube root; 5 is the cube root of 125.

The expressions $a^2, b^3, c^4, d^5 \ldots p^k$ are read respectively a square, b cube (or third), c fourth, d fifth . . . p kth.*

80. Model E.—Multiply $5x^2y$ by $3x^5y^4z^2$.

$5xxy . 3xxxxxyyyyzz$ is what we get for the factors of the

* Note the difference in meaning between a^2, a^3, a^4 and $a_1, a_2, a_3 \ldots$ (Read a two, a three, a four, etc.)

product. These can be rearranged so as to bring the numerical factors together, and bring all like letters together:

$$5 \times 3\ xxxxxxx\ yyyyy\ zz$$

which can be simplified thus:

$$15x^7y^5z^2.$$

EXERCISE XXXIX.

1. Multiply $3x$ by $4y$.
2. Multiply $3xy$ by $7xy$.
3. Multiply $3abc$ by ac.
4. Multiply a^3 by a^2.
5. Multiply a^{17} by a.
6. Multiply $3a^2b^2$ by $4a^3b^2$.
7. Multiply $8a^4c$ by $5a^2bc^3$.
8. Multiply $12ab^4c^3$ by $15a^3bc^2$.
9. Multiply $7a^5c^7$ by $4a^2bc^3$.
10. Multiply a^8 by $3a^3$.
11. Multiply a^2x by ax^2.
12. Multiply $2abx$ by $7abx$.
13. Multiply xyz by $15x^2y^3z$.
14. Multiply $pqrs$ by $7pq^2r^3s^4$.
15. Multiply $19x^3yz$ by $111x^2y^4z$.
16. Multiply $13a^{13}$ by $7a^7$.
17. Multiply $301a^2x^2$ by $2a^{301}x^3$.
18. Multiply $117a^2h^2k^2$ by $11h^2k^2j^2$.
19. Multiply $211a^3x^2z$ by $112a^7x^9z^{11}$.
20. Multiply $71q^{17}r^7$ by $17q^7r$.

Multiply together the following expressions:

21. x^2y; xy; xy.
22. xyz; x^2yz; xy^2z; xyz^2.
23. a^3b^2c; a^2bc; ab^2c^3.
24. $3ax^2$; $5a^2xy$; $6ab^2x$.
25. $10h^2k$; $15hk^2$; $2h^3k^3$.
26. $2a^2bc$; $3a^3bc$; $4a^4bc$.
27. $11a^{11}bc$; $12ab^{12}c$; $5abc^5$.
28. $101a^{13}b^4c^3$; $111a^2b^{15}c^7$; $13a^{17}b^{23}c^{29}$.
29. $pqrs$; p^2qrs^2; $7pq^2r^3s^4$.
30. $301h^{301}$; $103h^{103}$; $101h^{101}k$.

81. When an expression is separated into two parts by a plus or a minus sign, the quantity is called a **binomial.**

When an expression is separated into three parts by plus or minus signs, the quantity is called a **trinomial.**

When an expression is separated into two or more parts by plus or minus signs, the quantity is called a **polynomial.**

When an expression is not separated into parts by plus or minus signs, the quantity is called a **monomial.**

The expressions multiplied in the preceding examples of this chapter are all monomials; the multiplication of binomials and other polynomials depends upon the multiplication of monomials.

82. Model F.—Multiply $5a^2$ by $(3x - 2a)$.

$$5a^2(3x - 2a) = 5a^2 . 3x - 5a^2 . 2a = 15a^2x - 10a^3.$$

EXERCISE XL.

Perform the following multiplications:

1. $7x(x - 1)$.
2. $3a^2(a - x)$.
3. $10h^2k(2h - 5k^2)$.
4. $11p^3q(3p^3q - 3pq^3)$.
5. $7xy^2(2xy^7 + 4x^7y)$.
6. $2(x - 5)$.
7. $3h(4k^2h^2 - 2h^2j^2)$.
8. $101h^3k^2(11h^{17}k^5 - 21h^2k^{15})$.
9. $38hij(23hji + 12ijh)$.
10. $10ax(100bx + 1000cx)$

Multiply together the following quantities:

11. $3;\ x - 5;\ 2$.
12. $3a(a - b);\ 2x;\ 3ax$.
13. $10x^2;\ 2y^2;\ \frac{1}{4}xy^2 + \frac{1}{5}x^2y$.
14. $3a^2b;\ a^2 - b^2;\ 2ab$.
15. $15pq^2;\ 9qr^2;\ \frac{1}{27}p^2r - \frac{1}{45}pr$.
16. $7ap^2;\ 4a^2p;\ 15bq^2;\ \frac{1}{20}ap - \frac{1}{21}bq$.

83. The pupil has noticed that where a polynomial is multiplied by any number, each term is multiplied; that is, $3(x - 5) = 3x - 15$.

84. This principle may be illustrated again as follows:*

A shopkeeper sends every week x dollars to the bank; his messenger uses every week \$1 for necessary expenses; his wife draws out every week y dollars for her personal use; his son remits every week z dollars, which is added to

* See § 12.

the shopkeeper's deposit. The increase of the shopkeeper's account each week, then, amounts to $x - 1 - y + z$. In 5 weeks the increase would be $5(x - 1 - y + z)$. But in 5 weeks the shopkeeper would send $5x$, the messenger would use up 5, the wife would draw out $5y$, and the son would remit $5z$. Therefore

$$5(x - 1 - y + z) = 5x - 5 - 5y + 5z.$$

85. This is the third of the important and fundamental principles of Algebra. It applies also to minus signs; e.g.,

$$-(x - z) = -x + z; \quad -\{x - y + z - a - b + c\}$$
$$= -x + y - z + a + b - c.$$

It may be stated as follows:

The product of a polynomial by a single term is found by multiplying each term of the polynomial successively.

The negative of a polynomial is found by taking the negative of each term successively.

86. The three principles stated on pp. 59, 60, and 72 are called the THREE FUNDAMENTAL LAWS, and they are known as

I. THE COMMUTATIVE LAW. (§ 57, I; § 58, I.)
II. THE ASSOCIATIVE LAW. (§ 57, II; § 58, II.)
III. THE DISTRIBUTIVE LAW. (§ 85.)

Upon these laws are based all the transformations of Algebra.

EXERCISE XLI.

Multiply the following expressions:

1. $5x(x - 5 + y)$; x^2y.
2. $3x^2y$; $2x^3y + 12x^2y^2 - 15xy^3 - y^4 - 4x^4$.
3. $-(x^2 + y^2 + 2xy)$; $3xy$.
4. $-2(3a^2b - 3ab^2 + a^3 - b^3)$; $4a^2$.
5. $7h^2k - 21h^3k + 42h^4k^2$; $5h^3k^2$.

6. $10pqr$; $5p^3 + 3q^3 + 7r^3 + 2p^2q + 3pq^2 + 4pq^3$.
7. $x^3 + 4x^2 + 5x - 24$; $2xy^3$.
8. $x^4 + ax^3 + bx^2 - cx - d$; $abcdx^4$.
9. $-3h^3(i^2k + k^2i + i^2j + j^2i + k^2j + j^2k)$.
10. $-4s^2t^3(7s^4 + 20s^3 - 15s^2 + 5t^2 + 10t^3 - 15t^4)$.

Distributive Factoring.

87. A glance will show when a monomial is a factor of every term of an expression, and such an expression can therefore readily be separated into factors. Thus in the expression

$$5a^2 - 15ab + 20a^3 - 5a$$

$5a$ is a factor of every term, and the whole expression is the product of

$$5a(a - 3b + 4a^2 - 1).$$

EXERCISE XLII.

Find the factors of the following expressions, or show that they are not factorable:

1. $ax + ay$.
2. $ax - ay$.
3. $5x + 10y$.
4. $3x - 15y$.
5. $a^2 - ab$.
6. $x^2 + x^2y$.
7. $x + x^2$.
8. $5x + 25x^3$.
9. $13x + 91y^2$.
10. $x^2y - xy^2$.
11. $7x + 49x^2 + 343x^3$.
12. $3x + 6xy + 2yz$.
13. $13x^3 + 65x^2y + 117xy^2$.
14. $343x^3y + 98x^2y^2 + 28xy^3 + 8y^4$.
15. $15x^2 + 9xy + 25y^2$.
16. $38x^3 + 57x^2y + 19xy^2$.
17. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4$.
18. $45x^2y^8 - 120x^3y^7 + 210x^4y^6 - 270x^5y^5$.
19. $n^2 - n + n^3 - n^4$.
20. $rs - 2r^2s + rst^2 + r^3s^3$.
21. $y^2 - 2mxy + x + y$.
22. $x^2 - 2xy + y^2$.
23. $x^3 - 3x^2y + 3xy^2$.
24. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.
25. $5xy^4 - 10x^2y^3 + 10x^3y^2 - 5x^4y$.

88. The process of "distributive factoring" is often used in rearranging terms of a long polynomial.

Model G. $x^3-ax^2-bx^2-cx^2+abx+acx+bcx-abc$

becomes, when we bracket like powers of x,

$$x^3-(ax^2+bx^2+cx^2)+(abx+acx+bcx)-abc,$$

and then, factoring each parenthesis,

$$x^3-x^2(a+b+c)+x(ab+ac+bc)-abc.$$

EXERCISE XLIII.

Rearrange by powers of x the first four expressions:

1. $x^4-a^2x^2+bx^2+cx^2-abx+acx+bc.$
2. $x^4+5x^3-4x^2-4ax+4b+ax^3+5ax^2-5bx-bx^2.$
3. $cx^4+nx^3+rx^2+crx+dr-mx^4+dx^3$
$-cmx^3+cnx^2-dmx^2+dnx.$
4. $2x^4+4x^3y+6x^2y^2+4xy^3+y^4-8x^3z+24x^2z^2$
$-32xz^3+16z^4.$
5. From $a^2x^2+b^2y^2+a^2y^2+b^2x^2+c^2x^2+c^2y^2$ make two groups, with x^2 and y^2 for coefficients.
6. From the same expression make three groups, with a^2, b^2, and c^2 for coefficients.
7. In the sum of the expressions in examples 1 and 3 find the coefficient of x^2.
8. In the expression $ax^3-bx^3+b^3x-a^3x+a^3b-ab^3$ find the coefficient of a^3 and that of b^3.
9. In $ax^2+2axk+ak^2+2hxy+2hky+2htx+2hkt$
$+by^2+2byt+bt^2+2gx+2gk+2fy+2ft+c$
find the coefficient of x and of y.
10. In the expression $a(bc-2ca)+b(ca-2)+c(ab-2bc)$ find the coefficient of a^2 and of a.

89. This sort of factoring is of especial use in the reduction of literal equations.

Multiplication of Polynomials.

90. In multiplying polynomials, the multiplicand, considered as one quantity, is multiplied by each term of the multiplier; then each of these partial multiplications is carried out.

Model H. $(5a+3b)(2a-7b)=5a(2a-7b)+3b(2a-7b)$
$=10a^2-35ab+6ab-21b^2=10a^2-29ab-21b^2.$

Model I. $(2a-5x)(3a-4x)=2a(3a-4x)-5x(3a-4x)$
$=6a^2-4ax-(15ax-20x^2)=6a^2-4ax-15ax+20x^2$
$=6a^2-19ax+20x^2.$

Model J. $(a+b-c)(a-b+c)=a(a-b+c)+b(a-b+c)$
$-c(a-b+c)=a^2-ab+ac+ab-b^2+bc-(ac-bc+c^2)$
$=a^2-ab+ac+ab-b^2+bc-ac+bc-c^2$
$=a^2-b^2+2bc-c^2.$

EXERCISE XLIV.

Multiply:

1. $2x-y;\ 2y+x.$
2. $p^2+3pq+2q^2;\ 7p-5q.$
3. $p^2-pq+q^2;\ p^2+pq+q^2.$
4. $h^2-hk+2k^2;\ h^2+hk-2k^2.$
5. $x^2+2xy+y^2;\ x^2+2xy-y^2.$
6. $x^2-4;\ x^2+5.$
7. $x^2-6x+9;\ x^2-6x+5.$
8. $x^2+5x-3;\ x^2-5x-3.$
9. $x+15;\ x-8.$
10. $x^3-3x+2;\ x^3-3x^2+2.$
11. $a^2+ab+b^2;\ a-b.$
12. $p^3-q^3;\ p^3+q^3.$
13. $h^2-k^2;\ h^4+h^2k^2+k^4.$
14. $x+y+z;\ x+y-z.$
15. $a^2+3b^2;\ a+4b.$
16. $x^3-2x^2+8;\ x+2.$
17. $a-2b+c;\ a+2b-c.$
18. $a^3-7a+5;\ a^2-2a+3.$
19. $3x-5y;\ 3x^2-4y^2.$
20. $20+x;\ x^2-10.$

Simplify these expressions:

21. $(3x^2+2x+1)(3x-5).$
22. $(x^2+5x+25)(x-5).$
23. $(x^2-7x-44)(x+11).$
24. $(3x^2+4x+5)(8x-10).$

25. $(6x^2 - 3x + 2)(2x + 1)$.
26. $(3x^2-2xy-y^2)(3x-y)$.
27. $(x^2+3xy+4y^2)(3x-4y)$.
28. $(x^3+2x^2+4x+8)(x-2)$.
29. $(4t^2 + 14t + 49)(2t-7)$.
30. $(81t^2-63t+49)(9t+7)$.
31. $(16-28p+49p^2)(4+7p)$.
32. $(38a^2-19y^2)(2a^2+y^2)$.
33. $(3x^2 + 2x - 5)(2x^2 + 5x - 1)$.
34. $(5x^3 + 4x^2y - 3xy^2)(5x^2 - 4xy + 3y^2)$.
35. $(x^3 + 3x^2 + 9x + 27)(x^2 - 2x - 3)$.
36. $(x^3 - 2x^2 + 4x - 8)(x^2 + x - 2)$.
37. $(x^2 + 3x + 9)(x - 3) - (x^2 - 2x + 4)(x + 2)$.
38. $(m^4 + n^2 - 1)^2 - (m^4 - n^2 + 1)^2$.
39. $(x^2 + 2x - 3)^2 - (x^2 - 2x + 3)^2$.
40. $(17a^2 - 16ab + 15b^2)^2 - (15a^2 + 16ab - 17b^2)^2$.

The Law of Signs.

91. It is convenient to notice a law in regard to the sign of the separate terms of the product.

Where any term of the multiplier is plus, the partial product obtained by it has for each of its terms the same sign as the corresponding term in the multiplicand; where any term of the multiplier is minus, the partial product obtained by it has for each term the sign opposite to that of the corresponding term in the multiplicand.

To express the four possible cases in a table:

Multiplicand.	+	+	−	−
Multiplier.	+	−	+	−
Product.	+	−	−	+

92. From this table the following law is easily derived:

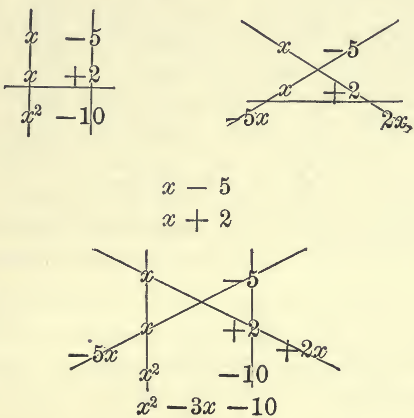
In multiplication like signs give plus and unlike minus.

93. The same rule holds for division, as will hereafter be shown.

CROSS-MULTIPLICATION.

94. Simple products like $(x - 5)(x + 2)$ have important features.

Model K.



The products of terms directly under each other are called STRAIGHT PRODUCTS ; of terms diagonally opposite each other CROSS PRODUCTS. The STRAIGHT PRODUCTS are not similar, and so CANNOT BE UNITED ; the CROSS PRODUCTS are similar and CAN BE UNITED.

In this example the straight products are x^2 and -10; the cross products are $-5x$ and $+2x$, and their sum $-3x$; the entire product is $x^2 - 3x - 10$.

EXERCISE XLV.

In the following examples name the straight products, the cross products, the sum of the cross products, and the entire product:

1. $(x+2)(x-7)$.
2. $(a+5)(a+9)$.
3. $(a+3)(a-10)$.
4. $(b+7)(b-6)$.
5. $(h+3)(h+8)$.
6. $(h-11)(h+2)$.
7. $(k-1)(k-4)$.
8. $(x+10)(x+10)$.
9. $(y-3)(y+3)$.
10. $(s+1)(s+99)$.
11. $(2x+1)(7x-1)$.
12. $(5a+1)(9a+1)$.
13. $(3a+2)(5a-2)$.
14. $(c+1)(2c+3)$.
15. $(k-7)(5k+1)$.
16. $(11-h)(2+h)$.
17. $(4b-1)(2b-3)$.
18. $(2x+3)(2x+3)$.
19. $(3y-7)(3y+7)$.
20. $(6s+1)(5s+10)$.
21. $(x+2y)(x-7y)$.
22. $(a+5b)(a+9b)$.
23. $(3a+2b)(5a+2b)$.
24. $(c+x)(3c+2x)$.
25. $(3k-4g)(5k+g)$.
26. $(10x-2h)(3x+5h)$.
27. $(4b-3k)(3b-2k)$.
28. $(3x+5y)(3x+5y)$.
29. $(3y-11z)(3y+11z)$.
30. $(6s+5t)(10s+7t)$.

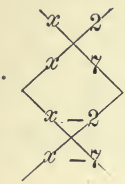
In each of the following examples ascertain what value x must have in order that both expressions may be equal; where the answers are not whole numbers, express them as common fractions:

31. $(x-5)(x-3)$; $(x-2)(x-7)$.
32. $(x-2)(x-8)$; $(x-3)(x-4)$.
33. $(x-7)(x+1)$; $(x+3)(x-5)$.
34. $(x+8)(x-5)$; $(x-10)(x-8)$.
35. $(x-2)(x+4)$; $(x+11)(x-4)$.
36. $(2x-5)(x-2)$; $(2x+5)(x-2)$.
37. $(2x-3)(x+12)$; $(2x+1)(x-11)$.
38. $(2x+7)(2x-5)$; $(4x-5)(x-5)$.
39. $(3x-5)(7x+2)$; $(3x-5)(7x+3)$.
40. $(17x-29)(x-3)$; $(17x-29)(x+2)$.

5. Products like those here formed by cross-multiplication are called **quadratic expressions.**

Factoring by Cross-Multiplication.

96. Model L.—In the product $x^2 - 9x + 14$ the straight products are x^2 and 14 and the sum of the cross products is $-9x$. The terms that give the straight products must have like signs, and therefore the cross products must have like signs; both cross products, then, are minus. To get the first straight product we have $x \times x$; to get the second 2×7; then the signs must be chosen so that both cross products are minus; that gives us $(x - 2)(x - 7)$.

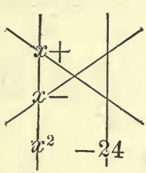


EXERCISE XLVI.

Name the straight products and the sum of the cross products in the following expressions, and find their factors:

1. x^2-5x+6.
2. x^2-2x+1.
3. x^2-3x+2.
4. x^2-4x+3.
5. x^2+4x+4.
6. x^2+5x+4.
7. $x^2+7x+12$.
8. x^2-7x+6.
9. $x^2+7x+10$.
10. $x^2-8x+15$.
11. x^2-8x+7.
12. $x^2+8x+15$.
13. $x^2-20x+19$.
14. $x^2-20x+64$.
15. $x^2+20x+36$.
16. $x^2+20x+51$.
17. $x^2-20x+75$.
18. $x^2+20x+84$.
19. $x^2-14x+49$.
20. $x^2-14x+13$.
21. $x^2-14x+33$.
22. $x^2-14x+24$.
23. $x^2+14x+45$.
24. $x^2+14x+48$.
25. $x^2-8x+16$.
26. $x^2-121x+120$.
27. $x^2-121x+238$.
28. $x^2-121x+1210$.
29. $x^2-121x+3550$.
30. $x^2-1000x+999$.

97. Model M.—In the product $x^2 - 5x - 24$ the straight products are x^2 and -24; $-5x$ is the sum of the cross products.



The terms giving x^2 for a product had like signs, and the terms giving -24 for a product had unlike signs; then the two cross products had unlike signs, and the minus cross product was the greater. To give x^2 for a

straight product we should have to multiply x and x; to give 24 we could multiply 1 and 24, 2 and 12, 3 and 8, 4 and 6; and in each case the larger number would have to be minus.

$$\begin{array}{c} x+1 \\ x-24 \\ \hline +x-24x \end{array} \qquad \begin{array}{c} x+2 \\ x-12 \\ \hline +2x-12x \end{array} \qquad \begin{array}{c} x+3 \\ x-8 \\ \hline +3x-8x \end{array}$$

Trying each in succession, the first pair of factors to give the right cross products is $(x+3)(x-8)$.

EXERCISE XLVII.

Name the straight products and the sum of the cross products in the following expressions, and wherever possible find their factors:

1. x^2-x-2.
2. x^2-x-6.
3. x^2-3x-4.
4. x^2+x-6.
5. $x^2+2x-15$.
6. $x^2+3x-28$.
7. $x^2-4x-21$.
8. $x^2-4x-32$.
9. $x^2+3x-70$.
10. $x^2+4x-96$.
11. $x^2-26x-407$.
12. $x^2-4x-437$.
13. $x^2+4x-221$.
14. $x^2-10x-231$.
15. $x^2-14x-6840$.
16. $x^2-28x-1100$.
17. $x^2+91x-900$.
18. $x^2+90x-1944$.
19. $x^2+x-10302$.
20. $x^2-54x-3627$.
21. $3x^2-x-10$.
22. $2x^2+3x-9$.
23. $5x^2-9x-2$.
24. $6x^2+x-15$.
25. $10x^2-9x-9$.
26. $15x^2-7x-2$.
27. $14x^2+11x-15$.
28. $21x^2+x-10$.
29. $10x^2-29x-21$.
30. $6x^2+17x-14$.

98. In the equation $2x-7=0$, x must have the value $\frac{7}{2}$; and if we substitute this value for x the expression $2x-7$ becomes zero.

Model N. $6x^2-5x-6$ factors into $(3x+2)(2x-3)$.
To make $3x+2=0$ we must have $x=-\frac{2}{3}$;
To make $2x-3=0$ we must have $x=\frac{3}{2}$.

EXERCISE XLVIII.

What value must we substitute for x in each factor of the following expressions in order that the factors may severally become equal to zero?

1. $x^2 - 3x + 2$.	6. $14x^2 + 5x - 1$.
2. $x^2 - 5x + 4$.	7. $21x^2 - 17x + 2$.
3. $6x^2 - x - 2$.	8. $3x^2 - 14x + 16$.
4. $2x^2 - 3x + 1$.	9. $12x^2 - 7x + 1$.
5. $x^2 - x - 2$.	10. $15x^2 - 11x - 14$.

99. This sort of factoring is of especial use in the solution of quadratic equations; it is also used with other devices in the reduction of fractional expressions.

Arrangement in Multiplication.

100. In multiplication it is often, if not always, convenient to arrange the work somewhat as in Arithmetic, with the partial products on separate lines, as follows·

Model O.

$$\begin{array}{r} x^2 - xy + 3y^2 \qquad\qquad \\ x - 2y \qquad\qquad\qquad\quad \\ \hline x^3 - x^2y + 3xy^2 \qquad\quad \\ - 2x^2y + 2xy^2 + 6y^3 \\ \hline x^3 - 3x^2y + 5xy^2 - 6y^3 \end{array}$$

Model P.

$$\begin{array}{r} x - 5 \qquad \\ x + 2 \qquad \\ \hline x^2 - 5x \qquad \\ + 2x - 10 \\ \hline x^2 - 3x - 10 \end{array}$$

101. The advantage of this arrangement is that similar terms in the different partial products may be more readily seen. A further help in this direction, for long examples, is the arrangement of multiplier and multiplicand "according to powers"; that is, taking for the first term of each that with the highest power of some one letter, for the next term the next power below, and so on; e.g.,—

$$45x^8y^2 + 210x^6y^4 - 10xy^9 + 210x^4y^3 + x^{10} - 120x^7y^3 \\ + 45x^2y^8 + y^{10} - 120x^3y^7 - 10x^9y - 252x^5y^5$$

could be arranged according to powers of x thus:

$$x^{10} - 10x^9y + 45x^8y^2 - 120x^7y^3 + 210x^6y^4 - 252x^5y^5 \\ + 210x^4y^6 - 120x^3y^7 + 210x^2y^8 - 10xy^9 + y^{10}$$

or according to powers of y thus:

$$y^{10} - 10xy^9 + 45x^2y^8 - 120x^3y^7 + 210x^4y^6 - 252x^5y^5 \\ + 210x^6y^4 - 120x^7y^3 + 210x^8y^2 - 10x^9y + x^{10}.$$

EXERCISE XLIX.

Rearrange the following expressions before multiplying:

1. $(a^2x^2 + x^4 + a^4)(a^4 + x^4 - a^2x^2)$.
2. $(x^2 + y^2 + xy)(y^3 + x^3)$.
3. $(a^6 + 3a^4b^2 + b^6 + 3b^4a^2)(a^6 - b^6 + 3a^4b^2 - 3a^2b^4)$.
4. $(a^5 - b^5 + 5ab^4 - 5a^4b + 10a^3b^2 - 10a^2b^3)(a^2 + b^2 - 2ab)$.
5. $(6x^2a^2 + x^4 + a^4 - 4x^3a - 4xa^3)(3a^2x - 3ax^2 - a^3 + x^3)$.
6. $(h^2s^2 + h^4 + s^4)(h^2 - s^2)$.
7. $(5y^2 - 2xy + 3x^2)(2xy + 3x^2 - 5y^2)$.
8. $(3hk^2 + h^3 - 3h^2k - k^3)(h^4 + k^4 + 6h^2k^2 - 4h^3k - 4hk^3)$.
9. $(21p^5q^2 + 35p^4q^3 + 35p^3q^4 + 7p^6q + p^7 + q^7 + 21p^2q^5 + 7pq^6)$ $(p^3 + q^3 + 3p^2q + 3pq^2)$.
10. $(x^4 - 6x^3y + 13x^2y^2 - 12xy^3 + 4y^4)(x^3 - x^2y + xy^2 - y^3)$.
11. $(x^2 + \frac{2}{3}x + \frac{1}{2})(\frac{2}{3} - \frac{1}{2}x + x^2)$.
12. $(3a^2 + \frac{1}{5} - 2a)(5a^2 - \frac{1}{2} - \frac{1}{3}a)$.
13. $(x^2 + xy + \frac{2}{3}y^2)(\frac{2}{3}y^2 - xy + x^2)$.
14. $(\frac{1}{2}x^2 - \frac{1}{3}x + \frac{5}{6})(\frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{5})$.
15. $(\frac{8}{27}a^3 + 2a^2b + \frac{27}{8}b^3 + \frac{9}{2}ab^2)(\frac{4}{9}a^2 + \frac{9}{4}b^2 + 2ab)$.

DIVISION.

102. In division we are given one factor of a number to find the other. The given factor is the divisor, the required factor is the quotient, and their product is the dividend. With regard to the signs of the separate terms of the quotient, they must be such that when multiplied by

the terms of the divisor they produce the signs given in the dividend.

To express the four possible cases in a table :

Divisor.	$+$	$+$	$-$	$-$
Dividend.	$+$	$-$	$+$	$-$
Quotient.	$+$	$-$	$-$	$+$

103. Hence the rule:

In division like signs give plus and unlike signs minus.

EXERCISE L.

Divide :

1. $12x^2$ *by* $2x$.
2. $24x^6$ *by* $4x^2$.
3. $9x^9$ *by* $3x^3$.
4. $4x^5$ *by* $5x^4$.
5. $38x^3a$ *by* $19ax$.
6. $400a^{20}b^3$ *by* $16ab$.
7. $72a^5b^6h^2k$ *by* $18a^3b^2hk$.
8. $325a^{17}z^5$ *by* $13a^{12}z^3$.
9. $(a+b)^5$ *by* $(a+b)^3$.
10. $72a^5(h^2-k^2)^{10}$ *by* $8a^3(h^2-k^2)^7$.
11. $abc \div (-c)$.
12. $15ah \div 3ah$.
13. $\dfrac{-39ax^2y^3}{13ax^2y^2}$.
14. $\dfrac{-348x^3y^5}{-12x^2y^2}$.
15. $\dfrac{-343a^3(a-b)^3}{-49(a-b)^2}$.
16. $\dfrac{-76(h^3-2k)^3}{19(h^3-2k)}$.
17. $333ijk \div (-9ik)$.
18. $-459a(b-c) \div 27a$.
19. $1002a(b-c)^3 \div \{-6(b-c)^2\}$.
20. $(-400) \div (-3.2)$.
21. $12a^2+6ab+30a$ *by* $6a$.
22. $2a+14ah+6a^2hk-26a^3$ *by* $2a$.
23. $21x^2y^3z^3-49x^3y^2z^3-9x^3y^3z^2$ *by* $3xyz$.
24. $3ij+6i^2k-9ki+12ijk$ *by* $3i$.
25. $-3ah^3k+6ah^2k^2+9ahk^3$ *by* $-3ahk$.
26. $14m^3n^4-42m^2n^3+28mn^2-35m^2n^2$ *by* $7mn$.
27. $18k^3(k+l)+9k^2(k^2+l)-12k(h+k)$ *by* $-3k$.
28. $2s(s+t)^3-6t(s+t)^3+8k(s+t)^3$ *by* $(s+t)^3$.

29. $r(r^2 + s^2) + s(r^2 + s^2) - t(r^2 + s^2)$ *by* $-(r^2 + s^2)$.

30. $a^2(u + v)^5 - a^2b^2(u + v)^3 + a^3c^3(u + v)^4$ *by* $a^2(u + v)^3$.

Long Division.

104. Model Q.—Divide $x^3 - y^3$ by $x - y$.

The correspondence of multiplication and division may be seen as follows. The arrangement given for division is recommended as the most compact.

Divisor.	$x - y$	Multiplicand.	$x - y$
Quotient.	$x^2 + xy + y^2$	Multiplier.	$x^2 + xy + y^2$
Dividend.	$x^3 - y^3$	1st partial product.	$x^3 - x^2y$
1st subtrahend.	$x^3 - x^2y$	2d partial product.	$x^2y - xy^2$
	$x^2y - y^3$	3d partial product.	$xy^2 - y^3$
2d subtrahend.	$x^2y - xy^2$	Product.	$x^3 - y^3$
	$xy^2 - y^3$		
3d subtrahend.	$xy^2 - y^3$		
	0		

EXERCISE LI.

Divide:

1. $x^4 - y^4$ *by* $x + y$.
2. $x^4 + x^2y^2 + y^4$ *by* $x^2 - xy + y^2$.
3. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ *by* $x^3 + y^3$.
4. $8c^4 - 22c^3x + 43c^2x^2 - 38cx^3 + 24x^4$ *by* $2c^2 - 3cx + 4x^2$.
5. $x^3 - 3x^2 + 9x - 27$ *by* $x^2 + 9$.
6. $a^3 + b^3 + c^3 - 3abc$ *by* $a + b + c$.
7. $s^6 - 2s^3 + 1$ *by* $s^2 - 2s + 1$.
8. $x^5 - x^3a^2 - x^2 + a^2$ *by* $x^2 - xa - x + a$.
9. $r^6 - r^3s^3 - r^3 + s^3$ *by* $r^3 + r^2 - r^2s + r - rs - s$.
10. $h^{10} + h^5g^5 + g^{10}$ *by* $h^2 + hg + g^2$.
11. $(\frac{1}{4}x^3 + 2) \div (\frac{1}{2}x + 1)$.
12. $(\frac{3}{8}a^3 - \frac{1}{3}ab^2 - {}^{1}b^3) \div (\frac{3}{2}a + b)$.
13. $(\frac{8}{9}a^4 + \frac{3}{2}ab^3 - \frac{9}{4}b^4) \div (a + \frac{3}{2}b)$.

14. $(\frac{1}{2}x^3 + \frac{2}{3}abx^2 - a^2b^2x - \frac{2}{3}a^3b^3) \div (\frac{1}{2}x + ab)$.

15. $(\frac{1}{6}a^4 + \frac{1}{3}a^2b^2 + \frac{3}{2}b^4) \div (\frac{1}{2}b^2 + \frac{1}{3}ab + \frac{1}{6}a^2)$.

Where there is a remainder in the following divisions, write the remainder as the numerator and the divisor as the denominator of a fraction which is one term of the complete quotient:

16. $(x^3 + y^3) \div (x - y)$.

17. $(x^{10} + x^5y^5 + y^{10}) \div (x^5 + y^5)$.

18. $\left(a^3 + a^2b + \frac{1}{3}ab^2 + \frac{b^3}{27}\right) \div \left(\frac{b}{3} - a\right)$.

19. $\{(x^2 - y^2)^2 + xy(x^2 + y^2 + 3xy)\} \div (x^2 + xy + y^2)$.

20. $(3a^4 + 27ab^3 - 10b^4) \div (a - 2b)$.

EXERCISE LII.

Perform the operations indicated in the following expressions:

1. $(a^3 + b^3)(a - b)(a^2 + ab + b^2)$.

2. $(a^2 + 2ab + b^2)(a^2 - 2ab + b^2)$.

3. $(a^3 + a^2b + ab^2 + b^3)(a - b)$.

4. $(a^2 + b^2)(a^2 - b^2) - (a - b)^4$.

5. $(a^3+3a^2b+3ab^2+b^3)(a+b)-(b-a)(a^3+a^2b+ab^2+b^3)$.

6. $[a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b + c)^3] \div (a + b + c)^2$.

7. $\dfrac{x^2(y + z) + y^2(z + x) + z^2(x + y) - x^3 - y^3 - z^3 - 2xyz}{x - (y - z)}$.

8. $\dfrac{h^2(k + s) + k^2(s + h) + s^2(h + k) + 3hks}{ks + sh + hk}$.

9. $\dfrac{(s^2 + st + t^2)(s^2 - st + t^2)(s^2 - t^2) - (3s^4t^2 - 3s^2t^4)}{(s - t)^3}$.

10. $\dfrac{(h^2 - 3h + 9)(h^2 + 5h + 25)(h^2 - 2h - 15)}{(h^3 - 8h^2 + 24h - 45)}$.

11. $\dfrac{x^{15} - y^{15}}{(x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x^2 + xy + y^2)(x - y)}$.

12. $\dfrac{(x^2 - y^2)^2 + 3x^2y^2}{x^2 - xy + y^2}$. 13. $\dfrac{a^3 - 3a^2b + 3ab^2 - b^3 + q^3}{a - b + q}$.

14. $\dfrac{(a^4+4a^3b+6a^2b^2+4ab^3+b^4)(a^2-2ab+b^2)}{a^4-2a^2b^2+b^4}$.

15. $\dfrac{(x^3-1)^2}{x^2-2x+1}$.

EXERCISE LIII.

Substitute:

1. $y=x+2$ in the expression $3x+4y-25$.
2. $z=2y-3$ in the expression $3x+3z-21$.
3. $z=3-y$ in the expression $z+10-5y$.
4. $y=2x+3$ in the expression $x^2+3xy+y^2$.
5. $x=y-1$ in the expression $3x+2xy+2x^2$.
6. $z=1-y$ in the expression $y^2+z^2+3xy+y+z$.
7. $y=2-3x$ in the expression $x(y-3)-y(x-3)+x^2+y^2$.
8. $x=2y+7$ in the expression $x^2+xy+2y^2$.
9. $y=-x^2$ in the expression $xy+1$.
10. $y=x^2+x+1$ in the expression $y+xy-x^3$.
11. $a=x^3-12x+16$; $b=x^3-12x-16$; $c=x^2-16$ in $\dfrac{ab}{c}$.
12. $h=x^3-2x+1$; $k=x^3-3x^2+3x-1$ in $\dfrac{h(x^3-3x+2)}{k}$.
13. $b=a^3+x^3$; $c=a^2+ax+x^2$ in $\dfrac{b(a^2+ax+x^2)}{c(a^2-ax+x^2)}$.
14. $s=x^4-4x^3a+6x^2a^2-4xa^3+a^4$; $t=x^4-2x^3a+2xa^3-a^4$ in $\dfrac{s(x^2+2ax+a^2)}{t}$.
15. $x=a+b+c$; $y=(ab+bc+ca)$ in $\dfrac{xy-abc}{a+b}$.
16. $s=a^2$; $t=bc$ in $\dfrac{(s-t)^3+8t^3}{s+t}$.
17. $x+3$ for z in the equation $4z^2+9y^2=24z$.
18. $x-5$ for z in the equation $z^2=10z-y^2$.

19. $z + 5$ for y in the equation $16x^2 - 25y^2 + 200xy = 0$.

20. $h - 1$ for x in the equation $3x^2 - 2xy = 2y - 2hy$.

21. $a - b$ for y in the equation $2a^2 - y^2 = y(a+b) + 2ab$.

In each of the following examples, what value must x have in order that the given expressions may be equal to zero?

22. $\dfrac{x^3 - 11x + 14}{x^2 + 2x - 7}$.

23. $\dfrac{x^4 - 14x^2 + 17x - 6}{x^3 + 3x^2 - 5x + 2}$.

24. $\dfrac{x^4 - 4x^3 - 11x + 44}{x^3 - 11}$.

25. $\dfrac{3x^5 - 16x^4 - 35x^3 + 2x - 14}{3x^4 + 5x^3 + 2}$.

26. $\dfrac{6x^5 - 15x^4 + 4x^3 + 4x^2 + x - 2}{6x^4 - 3x^3 - 2x^2 + 1}$

27. $\dfrac{6x^5 - 33x^4 - 10x + 55}{3x^4 - 5}$.

28. $\dfrac{45x^5 - 261x^4 - 260x^3 - 33x + 220}{15x^4 + 13x^3 - 11}$.

29. $\dfrac{57x^5 - 95x^4 + 63x^2 - 171x + 110}{19x^4 + 21x - 22}$.

30. $\dfrac{6x^5 - 4x^4 - 16x^3 + 10x^2 - 9x + 15}{2x^4 + 2x^3 - 2x^2 - 3}$.

31. $\dfrac{81x^5 + 306x^4 + 33x^3 - 6x - 22}{27x^4 + 3x^3 - 2}$.

CHAPTER IV.

IDENTITIES AND THEOREMS.

105. In regard to the following equations:

I. $3x + 5 = 2x + 7$ II. $3(x + 5) = 3x + 15$

the pupil will notice that while the first equation is true only for the particular value $x = 2$, the second is true for any value that may be chosen. Again, that the expression $3x + 5$ cannot, by any means we know of, be transformed into $2x + 7$; while $3(x + 5)$ is in a very simple way transformed into $3x + 15$.

106. These two kinds of equations have distinct names. An equation which is true only on condition that the letters in it have particular values is called an **equation of condition**; while an equation which is true for any values whatever of the letters in it is called an **identical equation.**

Equations of condition are the equations ordinarily met with in solving problems.

Identical equations (or equations of identity, or simply identities) may be recognized by the fact that one member can be transformed to the identical form of the other member. The two members are said to be identically equal, and sometimes the sign $\equiv$ is used instead of $=$.

The Proof of Theorems.

107. A Theorem is a general statement requiring proof. Theorems in algebra are often proved by stating them as

equations, and showing that one member can be transformed so as to be exactly like the other; that is, showing the equation to be identical.

108. Model A.—Prove the following theorem:

The product of the sum and difference of two numbers is equal to the difference of their squares.*

Proof.—Let a and b represent any two numbers, then the theorem is expressed by the following identity:

$$(a+b)(a-b) \equiv a^2 - b^2.$$

In transforming the first member we get for straight products a^2 and $-b^2$; the cross products are ab and $-ab$, and their sum is zero; so that the entire product is $a^2 - b^2$.

EXERCISE LIV.

Prove the following theorems:

1. The square of the sum of two numbers is equal to the square of the first number, plus twice the product of the two, plus the square of the second.

(The identity is $(a+b)^2 \equiv a^2 + 2ab + b^2$.)

2. The square of the difference of two numbers is equal to the square of the first number, minus twice the product of the two, plus the square of the second.

3. The square of any polynomial is equal to the sum of the squares of the separate terms, added to twice their products, taken two at a time.

(The straight products are the squares; show that the cross products are double.)

4. The difference of the squares of two consecutive numbers is one more than double the less number.

(Let a be the less number, $a+1$ the greater.)

5. The difference of the squares of two consecutive numbers is equal to their sum.

*This theorem will be hereafter referred to as THEOREM A.

6. The sum of the squares of two consecutive numbers is one more than twice their product.

7. The difference of the cubes of two consecutive numbers is one more than three times their product.

8. The sum of the cubes of two numbers, divided by the sum of the numbers, is equal to the sum of the squares of the two numbers minus the product of the numbers.

9. The difference of the cubes of two numbers, divided by the difference of the numbers, is equal to the sum of the squares of the two numbers plus the product of the numbers.

10. The product of three consecutive numbers is equal to the difference between the middle number and its cube.

(Let $a-1$, a, and $a+1$ be the numbers.)

11. The product of two consecutive numbers is equal to the smaller number plus its square.

12. Prove Theorem A, letting a be the smaller of the two numbers, and b the difference between them.

13. Prove the same theorem, letting a be the larger of the two numbers, and b the difference between them.

14. The difference of the squares of two consecutive even numbers is twice their sum.

15. The difference of two numbers formed by the same two digits in opposite order is always divisible by 9.

16. The difference of two numbers formed by the same three digits in opposite order is always divisible by 99.

109. Identities may be translated into theorems.

Model B.—In the identity $(a+b)^3 \equiv a^3+b^3+3ab(a+b)$, a and b stand for any two numbers, because an identity is true for all numbers; $a+b$ is the SUM, a^3 and b^3 are the cubes, ab is the PRODUCT of any two numbers. So the theorem may be stated:

The cube of the sum of any two numbers is equal to the

sum of their cubes plus three times their product multiplied by their sum.

Similarly for the identity

$$(a-b)^3 \equiv a^3 - b^3 - 3ab(a-b).$$

Many algebraic theorems are so complicated that they can be conveniently stated only as identities. Such are the following.

EXERCISE LV.

Show that these identities are true:

1. $(a^2+ab+b^2)(a^2-ab+b^2) \equiv a^4+a^2b^2+b^4.$
2. $a^5+b^5 \equiv (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4).$
3. $a^5-b^5 \equiv (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4).$
4. $a^4-b^4 \equiv (a^2+b^2)(a+b)(a-b).$
5. $(a+b+c)^3 \equiv a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3.$
6. $(a+b)^4 \equiv a^4+4a^3b+6a^2b^2+4ab^3+b^4.$
7. $(a+b+c)(bc+ca+ab) \equiv a^2(b+c)+b^2(c+a)$ $+c^2(a+b)+3abc.$
8. $x^6+y^6 \equiv (x^2+y^2)(x^4-x^2y^2+y^4).$
9. $x^6-y^6 \equiv (x+y)(x-y)(x^2-xy+y^2)(x^2+xy+y^2).$
10. $a^{15}+b^{15} \equiv (a^3+b^3)(a^{12}-a^9b^3+a^6b^6-a^3b^9+b^{12})$ $\equiv (a^5+b^5)(a^{10}-a^5b^5+b^{10}).$

APPLICATIONS OF THEOREM A.

110. Theorem A is by far the most important of those here given. Its application leads to one or two convenient processes in Arithmetic. In Algebra its applications, to be hereafter shown, are of even greater importance.

Finding Squares by Theorem A.

111. By Theorem A the square of any number can be readily calculated when the square of any number near it is known.

Model C.—To find the square of 23:

$$\begin{aligned} 23 + 20 &= 43 \\ 23 - 20 &= 3 \\ \hline 129 &= 23^2 - 20^2 \\ \therefore\ 23^2 &= 529 \end{aligned}$$

Model D.—To find 59^2:

$$\begin{aligned} 60 + 59 &= 119 \\ 60 - 59 &= 1 \\ \hline 119 &= 60^2 - 59^2 \\ \therefore\ 3481 &= 59^2 \end{aligned}$$

EXERCISE LVI.

Find in this way the squares of the following numbers:

1. 45.	3. 78.	5. 29.	7. 249.	9. 988.
2. 37.	4. 117.	6. 198.	8. 1011.	10. 793.

A somewhat different method is as follows:

Model E.—To find the square of 23:

$$\begin{aligned} 23 + 3 &= 26 \\ 23 - 3 &= 20 \\ \hline 520 &= 23^2 - 3^2 \\ \therefore\ 529 &= 23^2 \end{aligned}$$

Model F.—To find 59^2:

$$\begin{aligned} 59 + 9 &= 68 \\ 59 - 9 &= 50 \\ \hline 3400 &= 59^2 - 9^2 \\ \therefore\ 3481 &= 59^2 \end{aligned}$$

In applying this theorem the pupil must try to find some number which, added to or subtracted from the given number, will make an easy multiplier.

EXERCISE LVII.

By the second method find the square of:

1. 111.	3. 98.	5. 85.	7. 1998.	9. 288.
2. 57.	4. 52.	6. 49.	8. 311.	10. 63.

EXERCISE LVIII.

Find by Theorem A the difference of the squares of each of the following pairs of numbers:

1. 3 and 73.	6. 121 and 120.
2. 9 and 109.	7. 8133 and 8131.
3. 575 and 425.	8. 2731 and 269.
4. 339 and 319.	9. 101 and 99.
5. 1723 and 277.	10. 10001 and 1.

The pupil may with profit invent arithmetical illustrations for the other theorems.

FACTORING BY THEOREM A.

112. Wherever, in the product of two binomial factors, the cross products disappear, the factors will be seen to be, respectively, the sum and the difference of the same two quantities, or multiples of them; so that such a product is also an illustration of Theorem A.

Model G.—For example, $18x^2 - 8$ is the product of $3x - 2$ and $6x + 4$, which may also be written $3x - 2$ and $2(3x + 2)$; and in fact the original expression might have been written $2(9x^2 - 4)$ and the parenthesis factored by Theorem A: since it is the difference of the squares of $3x$ and 2, it is the product of their sum $(3x + 2)$ and their difference $(3x - 2)$.

EXERCISE LIX.

Factor the following expressions:

1. $4x^2 - 9$.
2. $8x^2 - 162$.
3. $18x^2 - 32$.
4. $27x^2 - 147$.
5. $1728x^2 - 12$.
6. $49x^2 - 16y^2$.
7. $4a^2b^2 - 121x^2$.
8. $36x^2y^4 - 25z^6$.
9. $100h^4 - 36k^2$.
10. $1210a^3b^3 - 10ab$.
11. $243xy^2z^3 - 12x^3y^2z$.
12. $a^2b^2 - 9c^2$.
13. $a^6b^6 - 49$.
14. $1 - x^2y^2$.
15. $4 - a^2$.
16. $9 - 16x^2y^4$.
17. $1 - 100h^6k^2$.
18. $75x^{10} - 48a^8$.
19. $9a^2b^4c^6 - 9x^{16}$.
20. $1 - 100a^6b^4c^2$.

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21. $(z + y)^2 - x^2$.
22. $(z - y)^2 - x^2$.
23. $(x + y)^2 - 4$.
24. $(x + 1)^2 - a^2$.
25. $(x + 2y)^2 - z^2$.
26. $(2x + 3a)^2 - 9y^2$.
27. $(x + 6a)^2 - 4y^4$.
28. $(x^2 + 7)^2 - 49y^2$.
29. $(2x - 5y)^2 - 9c^2$.
30. $(9x - a^2)^2 - a^2$.
31. $x^2 - (y + z)^2$.
32. $x^2 - (y - z)^2$.
33. $4 - (x + y)^2$.
34. $a^2 - (x + 1)^2$.
35. $z^2 - (x + 2y)^2$.
36. $9y^2 - (2x + 3a)^2$.
37. $4y^4 - (x + 6a)^2$.
38. $49y^2 - (x^2 + 7)^2$.
39. $9c^2 - (2x - 5y)^2$.
40. $a^2 - (9x - a^2)^2$.

What relation exists between the expressions numbered **21** *and* **31, 22** *and* **32, 23** *and* **33**, *etc.? What relation exists between their factors?*

41. $(a + b)^2 - (x + y)^2$.
42. $(a - b)^2 - (x - y)^2$.
43. $(x - y)^2 - (a - b)^2$.
44. $(a - h)^2 - (b + k)^2$.
45. $(2x - y)^2 - (x^2 + 2z)^2$.
46. $(a - 3x)^2 - (4y - z)^2$.
47. $(a + 2b)^2 - (4x + 5y)^2$.
48. $(a^2 + b)^2 - (3a + b^2)^2$.
49. $(x^3 - x^2)^2 - (x - 1)^2$.
50. $(a^4 - a^3b)^2 - (a^2b^2 - ab^3)^2$.
51. $x^2 + 2xy + y^2 - (a + b)^2$.
52. $a^2 - 2ab + b^2 - (x - y)^2$.

53. $9a^2 - 6ab + b^2 - (2x - 3y)^2$.
54. $a^2 - 2ah + h^2 - (x^2 + 2z)^2$.
55. $a^4 + 2a^2b + b^2 - (4y - z)^2$.
56. $a^2 - (b^2 - 2bc + c^2)$.
57. $4a^2 - (y^2 - 2yz + z^2)$.
58. $c^2 - (25a^2 - 30ab + 9b^2)$.
59. $(x - y)^2 - (a^2 + 6ab + 9b^2)$.
60. $x^2 + 10ax + 25a^2 - (16z^2 + 8z + 1)$.
61. $9x^2 - 4a^2 + 12ab - 9b^2$.
62. $1 - a^2 - 2ab - b^2$.
63. $16x^2y^2 - x^2 - 14xy - 49y^2$.
64. $a^2 - 2ab + b^2 - x^2$.
65. $9c^2 - 4a^2 + 4ab - b^2$.
66. $x^2 + 2xy + y^2 - a^2 + 2ab - b^2$.
67. $a^2 - 2ab + b^2 - c^2 - 2cd - d^2$.
68. $x^2 - 4ax + 4a^2 - b^2 + 2by - y^2$.
69. $4x^2 - 12bx + 9b^2 - 9x^4 + 30b^2x^2 - 25b^4$.
70. $x^2 + 2x + 1 - a^4 - 2a^2b^2 - b^4$.

113. In the ten examples last preceding the groups of terms that must be bracketed together to make a perfect square are easily found; but even where carelessly arranged they may be picked out by selecting the cross products first, then the straight products ; and by remembering that in a perfect square the straight products are $+$, and consequently the negative straight products must appear in the negative brackets, i.e. the subtrahend, the second of the two groups into which the expression is to be divided.

Model H.—In the expression $x^4 - x^2 - 9 - 2a^2x^2 + a^4 + 6x$ $2a^2x^2$ and $6x$ are the cross products, and the two groups are $x^4 - 2a^2x^2 + a^4$ and $- x^2 + 6x - 9$, of which the second must go in the $-$ bracket.

$$x^4 - 2a^2x^2 + a^4 - (x^2 - 6x + 9) \equiv (x^2 - a^2)^2 - (x - 3)^2.$$

The factors of this expression are

$$(x^2 - a^2 + x - 3)(x^2 - a^2 - x + 3).$$

EXERCISE LX.

Factor the following:

1. $x^2 + y^2 + 2xy - 4x^2y^2$.
2. $4a^2b^2 - 8ab - a^2 - 16b^2$.
3. $a^2 - 9a^2c^2 - 4ab + 4b^2$.
4. $x^2 - a^2 + 1 - 2x - 4ab - 4b^2$.
5. $6a - 8dx + 1 - x^2 - 16d^2 + 9a^2$.
6. $x^2 - a^2 + y^2 - b^2 - 2xy + 2ab$.
7. $4x^2 - 12ax - c^2 - k^2 - 2ck + 9a^2$.
8. $a^2 + 6bx - 9b^2x^2 - 10ab - 1 + 25b^2$.
9. $a^4 - 25x^6 + 8a^2x^2 - 9 + 30x^3 + 16x^4$.
10. $a^4 - a^2 + 4a^2 - b^2 + 2ab + 4$.

Some of the following examples have more than two factors:

11. $(a^2 - 3b^2)^2 - 4a^2b^2$.
12. $(2a^2 - 3b^2)^2 - a^2b^2$.
13. $(x^2 + 12)^2 - 49x^2$.
14. $(x^2 - 5x)^2 - 16b^2$.
15. $(h^2 - 5h)^2 - 36$.
16. $(6x^2 - 35y^2)^2 - x^2y^2$.
17. $(7x^2 - 11x)^2 - 36x^6$.
18. $(3x^2 - 10y^2)^2 - x^2y^2$.
19. $(3x^2 + xy)^2 - 100y^4$.
20. $(6h^2 - 17hk)^2 - 9k^2$.

Completing the Square.

114. If we know that an expression is the square of a binomial, we need know only two of its terms; the missing term can be constructed from those two by the use of Theorems 1 and 2.*

Model I.—Thus if we know that $9x^2$ and 49 are the first and last terms of a perfect square, we know that the terms of the binomial are $3x$ and 7; and the binomial must be then either $3x + 7$ or $3x - 7$; in the first case the middle term is $+42x$ (twice their product), and in the second case the middle term is $-42x$.

* See p. 89.

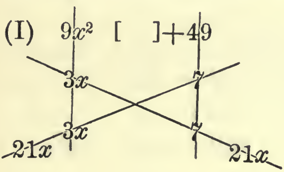
The complete square, then, of which $9x^2 + 49$ are the two straight products, would be either

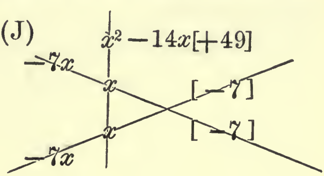
$$9x^2 + 42x + 49 \equiv (3x + 7)^2$$

$$\text{or} \qquad 9x^2 - 42x + 49 \equiv (3x - 7)^2.$$

Model J.—Again, if we know that $x^2 - 24x$ are the first two terms of a perfect square, the last can be found by remembering, according to Theorem 2, that while x^2 is the square of the first term of the binomial, $24x$ is twice the product of both terms; so the second term of the binomial is -12; and the last term of the square, then, would be 144. The complete square, then, of which $x^2 - 24x$ are the first two terms, is $x^2 - 24x + 144 \equiv (x - 12)^2$.

115. The same facts may be more clearly seen by considering the diagrams for cross-multiplication. In the first case the double cross product is missing; in the second case, since the middle term is known to be a DOUBLE cross product, the single cross products are each half of it.

(I)


(J)


EXERCISE LXI.

Complete the square implied in each of the following expressions:

1. $x^2 + 112x.$
2. $x^2 + 256.$
3. $x^2 - 144x.$
4. $x^4 + 16.$
5. $x^6 - 18x^3.$
6. $49x^2 + 625.$
7. $9x^2 - 24x.$
8. $64y^2 + 400y.$
9. $4x^2 + 729.$
10. $16x^6 - 200x^3y.$

116. We are sometimes enabled, by means of these facts, to transform an expression so that it can be treated

as the difference of two squares, and factored by Theorem A. Thus $64x^4 + 625$ becomes a perfect square by adding $400x^2$; and if we also subtract $400x^2$ we obtain the original expression unchanged in value, but transformed so as to be readily factored by Theorem A.

Model K.
$$\begin{aligned} 64x^4 + 625 &\equiv 64x^4 + 400x^2 + 625 - 400x^2 \\ &\equiv (8x^2 + 25)^2 - (20x)^2 \\ &\equiv (8x^2 - 20x + 25)(8x^2 + 20x + 25). \end{aligned}$$

EXERCISE LXII.

In the same way factor:

1. $x^4 + 4$.
2. $4x^4 + 81$.
3. $x^4 + 324$.
4. $4x^4 + 625y^4$.
5. $4x^4 + 1296z^8$.
6. $64x^4 + 81$.
7. $2500x^4 + 1$.
8. $324x^4 + 625$.
9. $1024x^4 + 81y^4$.
10. $4x^4 + 6561$.

117. Some trinomials can also be treated in this way. This is the case when the number to be added to the middle term to make it a double cross product is itself a perfect square.

Model L.—Factor $x^4 + x^2y^2 + y^4$.
$$\begin{aligned} x^4 + x^2y^2 + y^4 &\equiv x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &\equiv (x^2 + y^2)^2 - (xy)^2 \\ &\equiv (x^2 + y^2 + xy)(x^2 + y^2 - xy). \end{aligned}$$

EXERCISE LXIII.

In the same way factor:

1. $x^4 + 9x^2 + 81$.
2. $x^4 + x^2 + 1$.
3. $x^4 + x^2y^4 + y^8$.
4. $16x^4 + 4x^2 + 1$.
5. $1 + 9x^2 + 81x^4$.
6. $16x^4 + 36x^2 + 81$.
7. $x^8 + x^4y^4 + y^8$ (3 factors).
8. $256x^8 + 16x^4 + 1$.
9. $a^5 + 9a^3b^2 + 81ab^4$.
10. $48x^4 + 12x^2 + 3$.
11. $x^4 - 6x^2 + 1$.
12. $x^4 - 11x^2y^2 + y^4$.
13. $x^4 - 7x^2y^2 + y^4$.
14. $x^4 - 2x^2 + 1$.
15. $a^4 - 27a^2b^2 + b^4$.
16. $16a^4 - 24a^2 + 1$.
17. $81x^4 + 8x^2 + 16$.
18. $625x^4 + x^2 + 1$.

19. $x^4 - 66x^2 + 625.$

20. x^8+14x^4+625 (4 factors).

21. $x^5 - 11x^3 + x.$

22. $a^4b^2 - 27a^2b^4 + b^6.$

23. $80a^5 - 120a^3 + 5a.$

24. $243x^4y^2 + 24x^2y^2 + 48y^2.$

25. $2500x^4 + 4x^2 + 4.$

26. $3a^2x^4 - 51a^2x^2 + 48a^2.$

27. $80x^8 + 115x^6 + 405x^4.$

28. $7ax^4y^4 - 49ax^2y^4 + 567ay^4.$

29. $4x^4 + 162x^8 + 2.$

30. $4x^2 - 256x^4 - 1.$

Factoring Quadratics by Completing the Square.

118. Model M.—Factor $x^2 + 56x + 768.$

$$\begin{aligned} x^2 + 56x + 768 &\equiv x^2 + 56x + 784 - 16 \\ &\equiv (x + 28)^2 - 4^2 \\ &\equiv (x + 28 + 4)(x + 28 - 4) \\ &\equiv (x + 32)(x + 24) \end{aligned}$$

EXERCISE LXIV.

In the same way factor:

1. $x^2 - 62x + 945.$

2. $x^2 + 68x + 1155.$

3. $x^2 - 10x - 704.$

4. $x^2 - 50x + 616.$

5. $x^2 - 12x - 1260.$

6. $x^2 - 14x - 1176.$

7. $x^2 - 82x + 1512.$

8. $x^2 - 10x - 875.$

9. $x^2 - 8x - 768.$

10. $x^2 + 74x + 1344.$

11. $x^2 - 200x - 281600.$

12. $x^2 + 400x - 422400.$

13. $x^2 - 700x + 105600.$

14. $x^2 + 800x + 153600.$

15. $x^2 - 900x + 201600.$

16. $x^2 - 800x + 158400.$

17. $x^2 - 800x - 614400.$

18 $x^2 + 800x + 134400.$

19. $x^2 - 900x + 170100.$

20. $x^2 - 1400x - 633600.$

119. This process becomes somewhat more difficult when the coefficient in the middle term is an odd number.

Model N.

$$\begin{aligned} x^2 + 11x - 726 &\equiv x^2 + 11x + (\ ) - 726 - (\ ) \\ &\equiv x^2 + 11x + 30.25 - 756.25 \\ &\equiv (x + 5.5)^2 - (27.5)^2 \\ &\equiv (x + 5.5 + 27.5)(x + 5.5 - 27.5) \\ &\equiv (x + 33)(x - 22). \end{aligned}$$

EXERCISE LXV.

In the same way factor ·

1. $x^2 + 5x - 594.$
2. $x^2 - 9x - 792.$
3. $x^2 - 15x - 1134.$
4. $x^2 - 51x + 648.$
5. $x^2 + 81x + 1458.$
6. $x^2 - 75x + 1386.$
7. $x^2 - 75x - 13356.$
8. $x^2 + 87x + 1782.$
9. $x^2 + 67x + 1120.$
10. $x^2 - 79x + 1350.$
11. $x^2 + 83x + 1512.$
12. $x^2 - 87x + 1620.$
13. $x^2 - 99x + 2240.$
14. $x^2 - 3x - 5400.$
15. $x^2 + 129x + 3780.$
16. $x^2 - 101x + 2520.$
17. $x^2 - 57x - 2268.$
18. $x^2 + 135x + 4536.$
19. $x^2 + 145x + 5184.$
20. $x^2 - 21x - 4860.$
21. $x^2 - 21x - 1960.$
22. $x^2 + 117x + 3240.$
23. $x^2 - 119x + 2940.$
24. $x^2 - 19x - 2880.$
25. $x^2 + 165x + 6804.$
26. $x^2 - 139x + 4800.$
27. $x^2 + 37x - 2520.$
28. $x^2 + 139x + 3780.$
29. $x^2 - 99x + 1944.$
30. $x^2 + 9x - 6300.$

120. A still further complication is introduced when the x^2 term has a coefficient greater than 1.

Model O.—Factor $6x^2 - 11x - 72.$

The most convenient way to attack an example of this kind, if it is not practicable to factor it by inspection, is to divide the expression throughout by the coefficient of x^2, so as to reduce the expression to the form of those we have just been considering. It must not be forgotten that this divisor must be restored to the other factors when we find them.

$$x^2 - \frac{11x}{6} - 12 \equiv x^2 - \frac{11x}{6} + \left(\frac{11}{12}\right)^2 - 12 - \left(\frac{11}{12}\right)^2$$

$$\equiv x^2 - \frac{11x}{6} + \frac{121}{144} - \frac{1849}{144}$$

$$\equiv \left(x - \frac{11}{12}\right)^2 - \left(\frac{43}{12}\right)^2$$

$$\equiv\left(x-\frac{11}{12}+\frac{43}{12}\right)\left(x-\frac{11}{12}-\frac{43}{12}\right)$$

$$\equiv\left(x+\frac{32}{12}\right)\left(x-\frac{54}{12}\right)$$

$$\equiv\left(x+\frac{8}{3}\right)\left(x-\frac{9}{2}\right)$$

These factors, multiplied by the coefficient we divided by at first, will be the factors of the given expression.

$$\begin{aligned}6x^2-11x-72&\equiv 6(x+\tfrac{8}{3})(x-\tfrac{9}{2})\\&\equiv 3(x+\tfrac{8}{3}).2(x-\tfrac{9}{2})\\&\equiv(3x+8)(2x-9).\end{aligned}$$

Model P. $400x^2+678x-135$ is somewhat harder to factor by inspection. Dividing by 400,

$$x^2+\frac{678}{400}x-\frac{135}{400}\equiv x^2+\frac{678}{400}x+\left(\frac{339}{400}\right)^2-\frac{135}{400}-\left(\frac{339}{400}\right)^2$$

$$\equiv x^2+678x+\frac{114921}{160000}-\frac{168921}{160000}\equiv\left(x+\frac{339}{400}\right)^2-\left(\frac{411}{400}\right)^2$$

$$\equiv\left(x+\frac{339}{400}+\frac{411}{400}\right)\left(x+\frac{339}{400}-\frac{411}{400}\right)\equiv\left(x+\frac{750}{400}\right)\left(x-\frac{72}{400}\right)$$

$$\equiv\left(x+\frac{15}{8}\right)\left(x-\frac{9}{50}\right).$$

Whence the factors of $400x^2+678x-135$ are

$$400\left(x+\frac{15}{8}\right)\left(x-\frac{9}{50}\right)\equiv 8\left(x+\frac{15}{8}\right).50\left(x-\frac{9}{50}\right)$$

$$\equiv(8x+15)(50x-9).$$

121. Practice in the application of this method of factoring may be had by trying some of the harder examples of the preceding chapter. It is not well to spend much time on such examples as the following, because an easier

method will be given later, in which the factors can be found by a formula.

EXERCISE LXVI.

Factor the following by completing the square:

1. $30x^2 + 56x + 24$.
2. $21x^2 - 104x + 60$.
3. $36x^2 + 231x - 345$.
4. $200x^2 - 91x - 135$.
5. $210x^2 - 23x - 72$.
6. $24x^2 + 67x + 45$.
7. $36x^2 + 81x - 40$.
8. $150x^2 - 175x - 294$.
9. $48x^2 + 78x - 315$.
10. $400x^2 - 651x + 108$.

122. This method can be used to factor any * quadratic expression, even when the second number in the transformed expression is not a perfect square; because we can find the approximate square root of any number expressed in figures.

Model Q.—Factor $3x^2 + 8x + 1$.

$$x^2 + \frac{8x}{3} + \frac{1}{3} \equiv x^2 + \frac{8x}{3} + \frac{16}{9} - \frac{13}{9}$$

13.0000	3.605	$\equiv \left(x + \frac{4}{3}\right) - \left(\frac{\sqrt{13}}{3}\right)^2$
9		
400	66	
396	6	$\equiv \left(x + \frac{4}{3} + \frac{\sqrt{13}}{3}\right)\left(x + \frac{4}{3} - \frac{\sqrt{13}}{3}\right)$
40000	7205	
36025	5	$\equiv (x + 1.33 + 1.2)(x + 1.33 - 1.2)$
397500		$\equiv (x + 2.53)(x + .13)$

* An apparent exception occurs when the transformed expression becomes the SUM of two squares, instead of the difference. Thus $x^2 - 10x + 29 \equiv (x - 5)^2 + (2)^2$, to which Theorem A does not apply. This difficulty will be dealt with later.

EXERCISE LXVII.

In the following examples figure the coefficients of the factors to two places of decimals:

1. $x^2 + 2x - 1$.
2. $x^2 + 4x + 2$.
3. $x^2 + 6x + 7$.
4. $x^2 - 8x + 14$.
5. $x^2 - 16x + 61$.
6. $x^2 - 10x + 20$.
7. $x^2 - 20x - 20$.
8. $x^2 + 12x + 30$.
9. $x^2 + 8x - 1$.
10. $x^2 - 6x - 2$.
11. $x^2 + 3x + 1$.
12. $x^2 - 11x + 6$.
13. $x^2 - 7x + 11$.
14. $x^2 - 15x + 42$.
15. $x^2 + 5x + 3$.
16. $x^2 + 9x - 2$.
17. $x^2 - 13x - 7$.
18. $x^2 + 17x - 10$.
19. $x^2 - 21x - 20$.
20. $x^2 - 33x - 2$.
21. $x^2 + x - 1$.
22. $2x^2 + 3x - 4$.
23. $3x^2 + 2x - 2$.
24. $4x^2 + 9x - 3$.
25. $2x^2 + x - 1$.
26. $x^2 + 3x + 1$.
27. $3x^2 + 10x + 1$.
28. $5x^2 + 8x + 2$.
29. $2x^2 + 8x + 5$.
30. $4x^2 + 5x - 2$.

CHAPTER V.

FACTORABLE EQUATIONS.

123. A theorem has been defined as a general statement requiring demonstration; and an axiom as a general statement not requiring demonstration. One of the most important axioms in elementary algebra is the following:

124. The product of two or more factors can never be zero unless at least one of those factors is itself equal to zero.*

Two Answers to One Question.

125. Model A.—A square box 7 inches high has 160 square inches more in its lateral surface than on its bottom. What is the size of the bottom?

Let $x =$ length (and breadth) of the bottom.

Then $7x =$ area each side and $x^2 =$ area bottom.

① $28x - x^2 = 160$		
② $0 = 160 - 28x + x^2$		① $-28x + x^2$
③ $0 = (x - 8)(x - 20)$		② factored †
④ $0 = x - 8$	from	③ by Ax. A
⑤ $8 = x$		④ $+ 8$
⑥ $0 = x - 20$	from	③ by Ax. A
⑦ $20 = x$		⑥ $+ 20$

* This axiom will be hereafter referred to as AXIOM A.

† When an equation is so arranged that all its terms are on one side and zero on the other—in other words, so that we have an algebraic expression equated to zero—then the factors of that expression are sometimes loosely called the "factors of the equation."

126. This example has two answers, or, as we sometimes say, there are two values for x which satisfy the equation. This does not mean that the same quantity x can be equal to two different numerical expressions at the same time; the box cannot be 8 inches square and also 20 inches square. But a box 8 inches square would have the properties described in the equation, and so would a box 20 inches square; there are two sizes of square box that could be made so as to be 7 inches high and at the same time to have 160 square inches more in the lateral surface than on the bottom.

EXERCISE LXVIII.

The following examples have each two answers:

1. Both ends and one side of a rectangular field require 41 rods of fencing, and the area of the field is 200 square rods. Find its dimensions.

2. A field 17 rods long is made square by cutting off 72 square rods from one end. Find the width of the field.

3. A farmer trades grain with a seedsman for a basket of new seed, and agrees to fill a basket for him for every quart the basket contains; he filled the basket 12 times, and then found that he had given just one bushel too much. How many quarts did the basket hold?

4. A man contracted to pay 15 cents a foot for gilt picture-moulding and 20 cents per square yard for straw matting used in furnishing a square room in his house. He was astonished to find that the picture-moulding cost him $4 more than the matting. What was the size of his room?

5. In another room, where there were 2 yards more in the length than in the breadth, the moulding at 15 cents per foot cost $4.20 more than the matting at 20 cents per square yard. What were the dimensions of the floor?

6. There is a cylinder 8 inches high, whose lateral surface ($2\pi rh$) exceeds the area of its base (πr^2) by 88 square inches. Find its radius. (Use $\pi = 3\frac{1}{7}$.)

7. A square reception-room 11 feet high has walls and ceiling papered; 8 more rolls of paper (36 square feet to the roll) are required to paper the walls than are required for the ceiling; what is the size of the room?

8. A man sold a horse for $102 and found that his loss per cent was one-eighth of the number of dollars he had paid for the horse. How much had he paid?

9. A stock-raiser bought sheep for $210, but after he had lost 5 by sickness, he sold what he had left of the flock for $150, a loss of $1 a head. What did they cost him apiece?

10. A man bought 9 tons of coal in an inland town, and sold enough to get his money back at a profit of $4 per ton; afterwards the price rose $6 per ton, and he could have got his money back by selling 2 tons less. How much did he pay for his coal? How many tons did he sell?

127. If we represent by a the number which is given as 7 in the last illustrative problem, and by b the number which is given as 160, we can state five new problems by putting the following values for a and b in the statement of that illustrative problem:

$$a = 11;\quad b = 340 \qquad\qquad a = 8;\quad b = 60$$
$$a = 5;\quad b = 91 \qquad\qquad a = 9;\quad b = 160$$
$$a = 4;\quad b = 62$$

In the same way successive sets of figures are given for each of the ten problems in the last exercise:

Example 1.	(a)	27	42	34	39	148
($a = 41$; $b = 200$)	(b)	91	220	144	187	2400

Example 2.	(*a*)	23	30	28	40	44
($a = 17$; $b = 72$)	(*b*)	132	200	75	375	480
Example 3.	(*a*)	13	22	11	16	15
($a = 12$; $b = 4$ pecks)	(*b*)	5	5	3	$3\frac{1}{2}$	$4\frac{1}{2}$
Example 4.	(*a*)	10cts.	11cts.	8cts.	13cts.	25cts.
($a = 15$; $b = 20$;	(*b*)	15cts.	16cts.	18cts.	36cts.	$1
$c = \$4$)	(*c*)	$1.05	$2.30	56cts.	$1.20	$2
Example 5.	(*a*)	2 ft.	2 ft.	2 yds.	2 yds.	3 yds.
($a = 2$ yds.; b=15cts.;	(*b*)	10cts.	7cts.	10cts.	15cts.	11cts.
c = 20cts.; $d = \$4.20$)	(*c*)	21cts.	15cts.	18cts.	27cts.	36cts.
	(*d*)	$1.05	$1.08	$2.00	$3.00	$2.78
Example 6.	(*a*)	25	5	6	9	13
($a = 8''$; $b = 88$ sq. in.)	(*b*)	154	66	110	176	418
Example 7.	(*a*)	12	$10\frac{1}{2}$	9	10	6
($a = 11$; $b = 8$)	(*b*)	7	10	5	11	3
Example 8.	(*a*)	$144	$147	$105	$192	$105.60
($a = \$102$; $b = \$\frac{1}{8}$)	(*b*)	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{12}$	$\frac{1}{10}$
Example 9.	(*a*)	120	215	256	210	280
($a = \$210$; $b = 5$;	(*b*)	3	8	2	8	3
$c = \$150$)	(*c*)	85	140	210	135	225
Example 10.	(*a*)	10	12	50	30	32
($a = 9$; $b = \$4$;	(*b*)	3	2	1	4	3
$c = \$6$; $d = 2$)	(*c*)	4	3	5	2	2
	(*d*)	2	2	15	3	4

Both Answers Alike.

128. Model B.—In example 7 in the preceding exercise suppose the room had been 6 feet high, and that 4 more rolls of paper were required for the walls than for the ceiling.

Let x = the length (and breadth) of the room.

Then $6x$ = area of each side, and x^2 = area of ceiling.

① $24x - x^2 = 144$	
② $x^2 - 24x + 144 = 0$	① $- 24x + x^2$
③ $(x - 12)(x - 12) = 0$	② factored
④ $x - 12 = 0$	from ③ by Ax. A
⑤ $x = 12$	④ $+ 12$

Here, on account of a peculiar selection of figures, the two answers to the problem are alike. Similar results are obtained if, instead of 6 feet high and 4 more rolls, we say 9 feet high and 9 more rolls, or 12 feet high and 16 more rolls, or 15 feet high and 25 more rolls.

Try the effect, in problems 1, 2, 3, 4, and 6, of the numbers given below:

Example	1	2	3	4	6
(*a*)	60	18	16	20	7
(*b*)	450	81	2 bush.	24	154
(*c*)				$6	

Answers Apparently Different.

129. If we find two different numbers as the values of x in solving a quadratic equation, we naturally conclude that there are two answers thus implied for the problem which gave rise to the equation. We sometimes find, however, that the two answers thus indicated are the same.

Model C.—To divide 28 into two parts, whose product shall be 75.

Let x = one part; then $28 - x$ = the other.

① $x(28 - x) = 75$	(product of both parts)
② $28x - x^2 = 75$	same as ①
③ $x^2 - 28x + 75 = 0$	② $+ x^2 - 28x$
④ $(x - 25)(x - 3) = 0$	③ factored
⑤ $x - 25 = 0$	
⑥ $x = 25$	
⑦ $x - 3 = 0$	
⑧ $x = 3$	

Here the first answer means that one part can be 25; then the other must be 3; or that one part can be 3; then the other must be 25. The two answers are therefore really the same.

EXERCISE LXIX.

1. A rectangular field containing one acre* requires 924 feet of fencing. What are its dimensions?

2. Two cubical bins, side by side, extend the whole length of a 16-foot wall, and contain $9\frac{1}{2}$ cords* of kindling-wood. What is the size of each?

3. The number 30551 has two factors whose sum is 360. What are they?

4. A rectangular room requires 68 feet of picture-moulding, which runs above the top of the windows; and the same room requires 32 yards of carpet. What is its size?

5. A similar room requires 20 yards of border † half a yard wide, and 18 yards of carpet one yard wide. What is the size of the room?

6. A wharf which projects 27 feet into the water is made of two square platforms, and its total area is 377 square feet. What is its greatest width, and its least?

7. Forty-six rods of fencing are required for a field whose diagonal is 17 rods. What is the size of the field?

8. The government of Utopia once enacted that each liquor saloon should pay $10 for its own license, and the same amount for every other license granted in the same ward, and at the same time refused to grant licenses outside of the two central wards. Thus 48 licenses were granted, yielding a license tax of $12,800. How many licenses in each ward?

* An acre is 10 square chains, and a chain is 66 feet long; a cord of wood measures $4 \times 4 \times 8$ ft.

† Not including mitres.

9. Two years later the number of licenses had increased to 208, and the total tax had become \$217,040. How many then in each ward?

10. A rectangular corner lot and two square lots adjoining contain in all 9972 square feet, and have altogether 114 feet frontage on each street. All the lots on a street have the same depth. What are the dimensions of the lots?

11. Two numbers are reciprocals and their sum is $2\frac{4}{15}$. What are they?

12. Two men are separately hired to lay the curbstone on opposite sides of the same street, and receive for the whole job 8 days' pay between them; one third of the job was accomplished the first day. How many days did it take each man to do his own work?

13. By one pipe a tank is filled, and then by another immediately emptied, the whole operation requiring an hour all but 10 minutes; then both pipes are used to fill the tank again, which is done in 12 minutes. How long for each pipe?

Meaning of Negative Answers.

130. Sometimes one answer is negative and still can be interpreted as a reasonable answer. It is generally necessary to change the sense of some word, so as to have it mean just the opposite of its meaning in the statement of the problem.

Model D.—Two laborers applied for work on a farm, and were each sent to build a rod of fence, working by the hour. After they were through, the overseer decided that one of the men could build 3 rods a day more than the other, and that between them they had earned half a day's pay. How many rods could each build per day?

Let $x =$ the number of rods the poorer laborer builds in a day; then $x + 3 =$ the number of rods the better laborer builds in a day.

$$
\begin{array}{lll}
(1) & \dfrac{1}{x} + \dfrac{1}{x+3} = \dfrac{1}{2} & \\
(2) & 2x + 6 + 2x = x^2 + 3x & (1) \times 2x(x+3) \\
(3) & x^2 - x - 6 = 0 & (2) - 4x - 6 \\
(4) & (x-3)(x+2) = 0 & (3) \text{ factored}
\end{array}
$$

Whence we obtain $x = 3$ and $x + 3 = 6$; $x = -2$ and $x + 3 = 1$.

The answer $x = -2$ becomes intelligible if we assume that to build -2 rods means to destroy 2 rods, and that the destructive laborer has to pay by the hour for his amusement. While the better laborer is building his rod and earning thereby a day's pay, the other fellow gets through with destroying his rod and has to pay for the half-day that he uses up in so doing.

If we represent by a the number 3 in the above problem, and by b the number ^{1}, we can substitute the following numbers for a and b, and obtain in each case a new problem.

(a)	5	7	9	33	39
(b)	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{28}$	$\frac{1}{40}$

EXERCISE LXX.

1. A field 20 rods by 18 is made 352 square rods in area by cutting off a strip of uniform width on the side, and adding to it a strip of the same width on the end. What is the width of the strips?

2. If a fraction whose numerator is 70 be inverted and multiplied by 6 its new value will exceed its original value by 1. Find the fraction.

3. A boat which is able to go 5 miles an hour goes 21 miles southward in a certain river and then 24 miles northward, the whole time spent in travelling being 11 hours. How fast does the stream flow southward?

4. Both pipes at a certain water-tank are opened, and the tank, being empty at noon, is filled at 2 o'clock; but if the pipes were opened separately, one each day, the tank would be filled 3 hours later by one than by the other. At what time is the tank full in each case?

5. A manufacturer employs his son as superintendent at $3 per day, with the understanding that the average amount saved by him in the daily expenses of the factory is to be added to his daily wages. When the balance due to his savings has reached $400 on the books, he finds that the concern owes him, including his wages, five times the average daily saving of the factory. What is the average daily saving?

For these examples the following numbers will serve to state new problems:

Example 1.	(a)	20	16	25	32	35
$(a = 20;\ b = 18;$	(b)	13	10	22	30	12
$c = 352)$	(c)	230	105	540	880	342
Example 2.	(a)	8	72	16	45	80
$(a = 70;\ b = 6;$	(b)	4	36	144	900	16
$c = 1)$	(c)	3	5	7	11	15
Example 3.	(a)	3	7	6	13	8
$(a = 5;\ b = 21;$	(b)	12	34	30	62	22
$c = 24;\ d = 11)$	(c)	10	33	32	63	30
	(d)	8	10	19	10	8
Example 4.	(a)	3	6	4	6	8
$(a = 2;\ b = 3)$	(b)	8	5	15	35	63

Example 5.	(*a*)	4	5	2	6	10
($a = 3$; $b = \frac{1}{10}$;	(*b*)	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{8}$	5%	6%
$c = 400$; $d = 5$)	(*c*)	80	900	1152	3840	2500
	(*d*)	8	90	10	6	13

Answers Suggesting Related Problems.

131. A negative answer to a quadratic equation may not be interpretable as a reasonable answer to the problem which gave rise to the equation, but may suggest a new problem, closely related to the given one, and having the negative and the positive answers interchanged. Such is the following :

Model E.—A farmer bought a herd of cows for $400; if there had been 4 more in the herd, the price would have been $5 less apiece. How many were there?

Let $x =$ the number of cows in the herd.

$$① \ \frac{400}{x} = \frac{400}{x+4} + 5.$$

② $400x + 1600 = 400x + 5x^2 + 20x$ ①$\times x(x+4)$

③ $5x^2 + 20x - 1600 = 0$ ②$-1600-400x$

④ $x^2 + 4x - 320 = 0$ ③$\div 5$.

⑤ $x = 16$; $x = -20$.

Now we can change the algebraic sense of the problem by changing words like "more" to "less" and *vice versa.* The new problem would read :

A farmer bought a herd of cows for $400; if there had been 4 less in the herd they would have cost $5 more apiece. How many were there?

Let $x =$ the number of cows in the herd.

$$① \ \frac{400}{x} = \frac{400}{x-4} - 5$$

Whence $x = 20$ or -16.

EXERCISE LXXI.

Of the following problems, the negative answers suggest related problems in the opposite algebraic sense :

1. There are 40 rooms in a house, and 3 more rooms on a floor than there are floors. How many floors ?

2. Comparing a pail with a 140-quart tub, a man concluded that the tub held as many pailfuls as the pail held quarts. He found, however, that the tub held 4 pailfuls more. What was the capacity of the pail ?

3. In order to save 4 hours on his regular trip of 126 miles, a stage-driver finds it necessary to go 2 miles an hour faster than his usual rate. What is his usual rate ?

4. What is the price of eggs when 10 more in a dollar's worth lowers the price 4 cents a dozen ?

5. A rectangular field contains $1\frac{1}{2}$ acres; if its length is increased by 20 feet and its breadth diminished by 18 feet its area will be diminished 2340 square feet. What are its dimensions ?

For these examples the following numbers will serve to state new problems :

Example 1.	(a)	35	50	60	44	112
$(a = 40;\ b = 3)$	(b)	2	5	4	7	6
Example 2.	(a)	88	176	99	90	180
$(a = 140;\ b = 4)$	(b)	3	5	2	$4\frac{1}{2}$	3
Example 3.	(a)	6	12	4	2	5
$(a = 4;\ b = 126;$	(b)	176	135	120	180	150
$c = 2)$	(c)	3	4	$1\frac{1}{2}$	1	$2\frac{1}{2}$
Example 4.	(a)	5	2	25	50	20
$(a = 10;\ b = 4)$	(b)	1	1	8	2	5

Example 5.	(*a*)	2	3	1	4	$2\frac{1}{2}$
($a = 1\frac{1}{2}$; $b = 20$	(*b*)	24	14	12	-60	-45
$c = 18$; $d = 2340$)	(*c*)	20	30	10	-16	-30
	(*d*)	3120	7680	780	6240	-3600

Meaningless Answers.

132. In each of the following problems an answer will be obtained which satisfies the quadratic equation, but has no reasonable interpretation in the problem which gave rise to that equation; and the answer may not even suggest a related problem.

EXERCISE LXXII.

1. A square box 7 inches high and without a cover has 288 square inches of inside surface. What is the size of the bottom?

2. Of two large wheels, one makes 48 turns more than the other in rolling a mile; their tires, straightened out and laid end to end, would reach 21 feet. What is the circumference of each?

3. Two sprinters start opposite ways from the middle post of a straightaway 440-yard track; in one second they are 21 yards apart and one of them finishes his 220 yards 2 seconds ahead of the other. What is the speed of each?

4. A cistern containing 4 gallons is filled by one pipe and then immediately emptied by another, both operations requiring just 5 hours. One pipe delivers 3 gallons per hour more than the other. How many gallons from each pipe alone?

5. A man rows 5 miles down-stream and back in 4 hours. The stream flows 3 miles an hour. How fast can the man row in still water?

For these examples the following numbers will serve to state new problems:

Problem 1.	(a)	8	10	15	20	3
($a = 7$; $b = 288$)	(b)	185	176	544	249	$36\frac{1}{4}$
Problem 2.	(a)	40	10	88	96	88
($a = 48$; $b = 21$)	(b)	23	$32\frac{1}{2}$	22	$24\frac{3}{4}$	27
Problem 3.	(a)	400	300	1000	900	500
($a = 440$; $b = 21$;	(b)	18	22	$20\frac{1}{2}$	$20\frac{1}{4}$	$24\frac{1}{2}$
$c = 2$)	(c)	5	$2\frac{1}{2}$	$22\frac{1}{2}$	10	$\frac{5}{6}$
Problem 4.	(a)	6	8	10	4	21
($a = 4$; $b = 5$;	(b)	5	7	3	3	$13\frac{1}{2}$
$c = 3$)	(c)	7	9	7	5	5
Problem 5.	(a)	5	24	30	$25\frac{1}{2}$	$10\frac{1}{2}$
($a = 5$; $b = 4$;	(b)	9	14	12	10	5
$c = 3$)	(c)	4	5	6	7	2

133. The answers to a quadratic equation are not always commensurable; in that case they can be approximately stated as decimal fractions.

Model F.

1. $x^2 + 3x = 11$
2. $x^2 + 3x - 11 = 0$ — 1 $- 11$
3. $x^2 + 3x + \frac{9}{4} - \frac{53}{4} = 0$ — same as 2
4. $(x + \frac{3}{2})^2 - (3.64)^2 = 0$ — same as 3
5. $(x + \frac{3}{2} + 3.64)(x + \frac{3}{2} - 3.64) = 0$ — 4 factored
6. $(x + 5.14)(x - 2.14) = 0$ — same as 5

$$\left.\begin{array}{l} x = 2.14 \quad \text{or} \\ x = -5.14 \end{array}\right\} \textit{Ans.}$$

The following solutions are worthy of attention:

Model G.

1. $x^2 = 16$
2. $x^2 - 16 = 0$ — 1 $- 16$
3. $(x + 4)(x - 4) = 0$ — 2 factored

$$\left.\begin{array}{l} \text{4. } x + 4 = 0;\ x = -4 \\ \text{5. } x - 4 = 0;\ x = 4 \end{array}\right\} \text{3 Ax. A}$$

Hence there are two square roots to 16 (or to any arithmetical number); one is + and the other −.

Model H.

$$\begin{array}{ll} (1)\ x^2 = 3x & \\ (2)\ x^2 - 3x = 0 & (1) - 3x \\ (3)\ x(x - 3) = 0 & (2)\ \text{factored} \\ \left.\begin{array}{l} (4)\ x = 0 \\ (5)\ x - 3 = 0;\ x = 3 \end{array}\right\} & (3)\ \text{Ax. A} \end{array}$$

Notice that the value $x = 0$ satisfies the equation.

EXERCISE LXXIII.

1. What number divided into 81 will give itself for a quotient?

2. Three times the square of a number is equal to 10 times the number itself. What is the number?

3. A room is twice as long as it is wide, and its floor contains $144\frac{1}{2}$ square feet. How wide is it?

4. A square room 9 feet high takes twice as much paper to cover the walls as to cover the ceiling. What is the size of the ceiling?

5. In a current of 2 miles an hour a man takes 8 hours longer to row 24 miles up-stream than to row the same distance down. How fast can he row in still water?

6. A field 20 rods by 18 is made 353 square rods in area by cutting off a strip of uniform width on the side and adding a strip of the same width on the end. What is the width of the strips?

7. A square box 9 inches high has 300 square inches more on the sides than on the bottom. What are the dimensions of the box?

8. One man takes 4 hours longer than another to saw a cord of wood, and both, working together, can saw it in 3 hours. How long for each?

9. Two wheels, the sum of whose circumferences is equal to 8 yards, roll the same distance; and the number of turns

made by one wheel, added to the number of turns made by the other, gives a sum numerically equal to the number of yards traversed.* Find the circumferences of the wheels.

10. A line 100 centimetres long is divided into two parts, such that the longer part is contained in the whole line as many times as the shorter part is contained in the longer part. What is the length of each part?

A New Kind of Number.

134. The solution of certain quadratics, of which the simplest is $x^2 = -1$ or $x^2 + 1 = 0$, leads to an expression which has no interpretation according to the rules of algebra, so far as we have studied them at present. This expression is $\sqrt{-1}$, and it arises as follows:

Model I.

① $x^2 = -1$	
② $x^2 + 1 = 0$	① + 1
③ $x^2 - (-1) = 0$	② in form of $x^2 - y^2$
④ $x^2 - (\sqrt{-1})^2 = 0$	
⑤ $(x + \sqrt{-1})(x - \sqrt{-1}) = 0$	④ factored
⑥ $x + \sqrt{-1} = 0;\ x = -\sqrt{-1}$	⑤ Ax. A
⑦ $x - \sqrt{-1} = 0;\ x = \sqrt{-1}$	⑤ Ax. A

When we remember that two factors must be exactly alike in order that their product may be a perfect square, and also that two numbers which are alike in respect to sign can never have a negative product, we see that it is impossible for a negative quantity to have a square root like any other algebraic expression such as we have met hitherto. It is customary to represent this new and strange expression, $\sqrt{-1}$, by the small letter **i**, so that $\mathbf{i}^2 = -\mathbf{1}$.

* Have a letter for the number of yards traversed; also one other letter.

Then i^2 is a part of our algebraic system, and while we cannot say that i is, we will assume that it can be dealt with by ordinary algebraic laws.

Expressions in which i appears in the first degree are called **imaginary**.

Model I.—The solution of the equation $x^2 = -1$ now becomes:

(1) $x^2 = i^2$	
(2) $x^2 - i^2 = 0$	(1) $- i^2$
(3) $(x + i)(x - i) = 0$	(2) factored
(4) $x + i = 0;\quad x = -i$	(3) Ax. A
(5) $x - i = 0;\quad x = i$	(3) Ax. A

Model J.

(1) $x^2 + 16 = 0$	
(2) $x^2 - 16i^2 = 0$	Subst. $i^2 = -1$ in (1)
(3) $(x + 4i)(x - 4i) = 0$	(2) factored
(4) $x + 4i = 0;\quad x = -4i$	(3) Ax. A
(5) $x - 4i = 0;\quad x = 4i$	(3) Ax. A

Model K.

(1) $x^2 + 6x + 13 = 0$	
(2) $x^2 + 6x + 9 + 4 = 0$	same as (1)
(3) $x^2 + 6x + 9 - 4i^2 = 0$	subst. $i^2 = -1$ in (2)
(4) $(x + 3 + 2i)(x + 3 - 2i) = 0$	(3) factored
(5) $x + 3 + 2i = 0;\quad x = -3 - 2i$	(4) Ax. A
(6) $x + 3 - 2i = 0;\quad x = -3 + 2i$	(4) Ax. A

135. The great advantage of the substitution of i^2 at this stage in our study is to enable us to change the sum of two squares to the FORM of the difference of two squares, and thus to obtain a complete FORM of solution to any quadratic equation.

EXERCISE LXXIV.

Solve the following equations:

1. $-x^2 = 9.$

2. $6x - x^2 = 13.$

3. $20x = x^2 + 121.$

4. $\frac{3}{x} + x = 2.$

5. $\frac{2x - 7}{x} = x.$

6. $9x^3 + x^2 = x^2 + x.$

7. $x - 15 = \frac{61}{5 - x}.$

8. $x + 90\frac{1}{4} = \frac{x}{40}(6 - x).$

9. $\frac{1}{4x} + \frac{16 - x}{x} = \frac{10 - x}{4}.$

10. $\frac{18}{x - 2} + x = 8.$

Find algebraic solutions to the following problems:

11. Two numbers are reciprocals and their sum is $\frac{1}{2}$. What are the numbers?

12. Of two equal fractions the first has a numerator 5; its denominator exceeds the numerator of the second fraction by 7, and exceeds the denominator by 9. Find the two fractions.

13. What is the length of a rectangular garden, 29 square rods in area, which is enclosed by 20 rods of fencing?

14. In building a wall each laborer employed worked as many days as there were men on the job. If there had been 10 men less, the job would have taken 20 days. How many laborers were there?

15. A rectangular board 12 inches long can be made square by cutting 45 square inches off the end. How wide is it?

THE QUADRATIC FORMULA.

136. Any quadratic equation, if simplified as much as possible and so transposed as to give an expression equal to zero, will be in the form of

$$ax^2 + bx + c = 0,$$

which is called the **standard form** of the quadratic equation.

137. The letters a, b, and c are the three coefficients of the quadratic, and of course for any given set of values for these there will be two and only two answers. Thus for the following ten equations the values of a, b, and c are as given, and the answers may be found to be as in the table.

	a	b	c	*Answers.*
$x^2 - 5x + 6 = 0$	1	-5	6	$2;\quad 3$
$2x^2 - 3x - 2 = 0$	2	-3	-2	$2;\ -\frac{1}{2}$
$2x^2 - 5x + 2 = 0$	2	-5	2	$2;\quad \frac{1}{2}$
$6x^2 - 5x - 6 = 0$	6	-5	-6	$\frac{3}{2};\ -\frac{2}{3}$
$x^2 + 10x + 23 = 0$	1	10	23	$-3.586;\ -6.414$
$x^2 - 81 = 0$	1	0	-81	$9;\ -9$
$x^2 + 81 = 0$	1	0	81	$9i;\ -9i$
$9x^2 - 16 = 0$	9	0	-16	$\frac{4}{3};\ -\frac{4}{3}$
$9x^2 + 16x = 0$	9	16	0	$0;\ -\frac{16}{9}$
$x^2 + 2x + 10 = 0$	1	2	10	$-1+3i;\ -1-3i$

138. It will be shown now that these two answers to every quadratic may be obtained by the formulæ

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$

these are usually written, to save space, in one formula, with the so-called ambiguous sign, which can be read either plus or minus, each reading giving one of the two correct answers.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

139. This is called the **Quadratic Formula,** and is obtained as follows:

① $ax^2 + bx + c = 0$

② $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ ① $\div\, a$

③ $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$ from ② by completing the square

④ $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) = 0$ same as ③

⑤ $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$ ④ factored

⑥ $x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0;\quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

⑦ $x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0;\quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

⑥, ⑦: ⑤ Ax. A

Whence the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Solving Quadratics by the Formula.

140. Model L.

① $2x^2 = 3 - 3x$

② $2x^2 + 3x - 3 = 0$ ① $+ 3x - 3$

$a = 2$ $b^2 = 9$

$b = 3$ $4ac = -24$

$c = -3$ $b^2 - 4ac = 33$

$\sqrt{b^2 - 4ac} = 5.745$

$$x = \frac{-3 \pm 5.745}{4}$$

$$= \frac{2.745}{4} \text{ or } \frac{-8.745}{4}$$

$$= .686 \text{ or } -2.186$$

Model M.

① $\dfrac{360}{x} - 360x = 504x + 959$

② $360 - 360x^2 = 504x^2 + 959x$ ① $\times x$

③ $864x^2 + 959x - 360 = 0$ ② $+ 360x^2 - 360x$

$a = 864$ $b^2 = 919681$

$b = 959$ $4ac = -1244160$

$c = -360$ $b^2 - 4ac = 2163841$

$\sqrt{b^2 - 4ac} = 1471$

$$\begin{array}{r} 864 \\ 360 \\ \hline 5184 \\ 2592 \\ \hline 311040 \\ 1244160 \end{array}$$

$$x = \frac{-959 \pm 1471}{1728}$$

$x = \dfrac{512}{1728}$ or $x = -\dfrac{2430}{1728}$; which reduce to

$x = \dfrac{8}{27}$ or $x = -\dfrac{45}{32}$

EXERCISE LXXV.

*By means of the Quadratic Formula, find the two roots * to each of the following equations:*

(Commensurable roots.)	(Incommensurable roots.)
1. $8x^2 - 66x + 135 = 0.$	11. $2x^2 - 10x - 9 = 0.$
2. $36x^2 - 133x = 120.$	12. $13x^2 = 5x + 10.$
3. $72x^2 + 161x - 200 = 0.$	13. $21x^2 = 22x - 23.$
4. $54x^2 - 345x + 500 = 0.$	14. $100x^2 = 25x + 36.$
5. $180x^2 + 351x = 50.$	15. $x^2 + 3 = 2x + 1.$
6. $300x^2 - 385x + 73\frac{1}{2} = 0.$	16. $x^2 = x + 1.$
7. $480x^2 - 682x + 225 = 0.$	17. $10x^2 - 100x + 203 = 0.$
8. $128x^2 - 24x = 495.$	18. $64x^2 - 33x = 100.$
9. $320x^2 - 1114x + 675 = 0.$	19. $20x^2 = 30x + 41.$
10. $1056x^2 = 1610x + 1125.$	20. $100x^2 - 100x - 101 = 0.$

141. The two roots of a quadratic equation are sometimes represented by α and β, that is,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

EXERCISE LXXVI.

1. Prove that $\alpha + \beta = -\frac{b}{a}$.

2. Prove that $\alpha\beta = \frac{c}{a}$.

3. By means of examples 1 and 2 show that

$$a(x - \alpha)(x - \beta) = ax^2 + bx + c.$$

4. A quadratic equation whose first term is $6x^2$ has for answers $\frac{2}{3}$ and $-\frac{5}{2}$; what is the equation?

5. In the equation $3x^2 + 4x = 3$, find $\alpha - \beta$.

* The numerical value of x which satisfies an algebraic equation is called a ROOT OF THE EQUATION.

Factoring Equations of Higher Degrees.

142. Sometimes, but by no means always, equations of degree higher than the second can be factored; and in that case the factors lead to solutions by the same theory as ordinary quadratics.

Model N.

① $x^3 = 27$	
② $x^3 - 27 = 0$	① $-\ 27$
③ $(x - 3)(x^2 + 3x + 9) = 0$	② factored
④ $x - 3 = 0;\ x = 3$	③ Ax. A
⑤ $x^2 + 3x + 9 = 0$	③ Ax. A
⑥ $x = \dfrac{-3 \pm 3i\sqrt{3}}{2}$	⑤ Quad. Form.

Ans. $x = 3;\ x = 3\left(\dfrac{-1 + i\sqrt{3}}{2}\right);\ x = 3\left(\dfrac{-1 - i\sqrt{3}}{2}\right).$

These are the three cube roots of 27.

Model O.

① $x^4 - 14x^2 + 40 = 0$

② $(x^2 - 4)(x^2 - 10) = 0$

③ $(x + 2)(x - 2)(x + \sqrt{10})(x - \sqrt{10}) = 0$

④ $x + 2 = 0; \quad x = -2$

⑤ $x - 2 = 0; \quad x = 2$

⑥ $x + \sqrt{10} = 0; \quad x = -\sqrt{10}$

⑦ $x - \sqrt{10} = 0; \quad x = \sqrt{10}$

EXERCISE LXXVII.

1. Find the 3 cube roots of 64.
2. Find the 4 fourth roots of 81.
3. Find the 6 sixth roots of 64.
4. $x^4 - 34x^2 + 225 = 0.$
5. $36x^4 = 17x^2 + 144.$

6. $x^3 - 7x^2 = 18x.$
7. $(x^2 - 9)(3x^2 + 8x - 3) = 0.$
8. $(x^2 - 3x + 1)(x^2 - 3x - 2) = 0.$
9. $(x^2 - 2x)(x^2 - 5x - 10) = 4(x^2 - 2x).$
10. $3x^2 + 2x - 5 = 3x^3 + 2x^2 - 5x.$

Factoring by Parts.

143. This is an extension of distributive factoring, and sometimes serves as an elementary method for the solution of equations of higher degrees.

The factors of $Mx^2 - 3Mxy + 5My^2$ are

$$M(x^2 - 3xy + 5y^2).$$

This conclusion would be true whatever the size and shape of the letter M, and even if the letter M represented a different algebraic expression altogether. Thus the factors of $(3a - 7b)x^2 - 3(3a - 7b)xy + 5(3a - 7b)y^2$

are $$(3a - 7b)(x^2 - 3xy + 5y^2).$$

Here the letter M is replaced by the expression $(3a-7b)$. Now if in the multiplication $(3a - 7b)(x^2 - 3xy + 5y^2)$ the second factor is taken as the multiplier, the three expressions $(3a - 7b)x^2$, $-3(3a - 7b)xy$, and $5(3a - 7b)y^2$ are the partial products; distributive factors removed from them leave the multiplier, which is seen to be the same in all of the partial products.

Model P.—In the expression

$$x^3 - 5x^2 - 9x + 45$$

the first two terms and the last two terms may be taken as partial products:

$$x^3 - 5x^2 + (-9x + 45) \equiv x^2(x - 5) + 9(-x + 5).$$

If the sign before the second partial product is changed the multiplicand will appear the same in each partial product :

$$\begin{aligned} x^2(x-5)+9(-x+5) &\equiv x^2(x-5)-9(x-5) \\ &\equiv (x-5)(x^2-9). \end{aligned}$$

The same expression may be factored by grouping the first term with the third, and the second with the fourth, to form the two partial products:

$$\begin{aligned} x^3-5x^2-9x+45 &\equiv x^3-9x-5x^2+45 \\ &\equiv x(x^2-9)-5(x^2-9) \\ &\equiv (x^2-9)(x-5). \end{aligned}$$

Of course the prime factors of this expression would be

$$(x+3)(x-3)(x-5).$$

EXERCISE LXXVIII.

*Factor by parts :**

1. $ax+by-ay-bx.$
2. $2x+xy-2y-x^2.$
3. $x^2-16+ax-4a.$
4. $ax+4x^2+4a+16x.$
5. $x^2+5x+6-x^3-2x^2.$
6. $x^3-3x^2-25x+75.$
7. $x^3+10x^2-81x-810.$
8. $112+16x-7x^2-x^3.$
9. $99-9x-11x^2+x^3.$
10. $121+121x-x^2-x^3.$
11. $(2x-y)(4x^2+2xy+y^2)-(2xy+y^2)(2x-y).$
12. $(1-2y)(1+2y+4y^2)-(1-2y)^2.$
13. $y^2(1-4y+4y^2)-(1-2y)^2.$
14. $a^2(6-a)-(36-a^2).$
15. $(10y-1)100y^2+(100y^2-10y)+(10y-1).$
16. $(a+b+c)^2+3a+3b+3c.$
17. $(x-17)(x+7)+x(x^2-49).$
18. $a^2+ab+ac+bc.$
19. $ax+by+bx+ay.$

* It may be necessary in some cases to rearrange the terms.

20. $6a + b^2 + 6b + a^2 + 2ab.$
21. $ax + bz - (bx + az).$
22. $5ax + by - (5ay + bx).$
23. $pr - qs + qr - ps.$
24. $f^2x^2 - ag^2 - (af^2 - g^2x^2).$
25. $x^2y^2 + x^2 - (y^2 - x^3).$
26. $x^2 + x - 2y(2y + 1).$
27. $a^2x + aby + a^2y + abx.$
28. $2ax + 3by - (2ay + 3bx).$
29. $x^4 - 3y^3 + (3x^2y^2 - x^2y).$
30. $3ax^2 + (x + y)(x - y) - 3axy.$
31. $ax + 3x^2y(x - 5y) - 5ay.$
32. $xy - 9 + x^2 + 3y.$
33. $ab + a^2b^2 - (x^2 + x^4).$
34. $a^2 + 3x + a - 9x^2.$
35. $hy + (h + k)^2 + ky.$
36. $ab(b^2 - a^2) + 3ab(ab - h^2) - h^2(b^2 - a^2).$

Model Q.—Solve the equation:

(1) $x^4 - 6x^2 + 8 = x^3 - 4x$
(2) $x^4 - 6x^2 + 8 - x^3 + 4x = 0$ (1) $- x^3 + 4x$
(3) $(x^2 - 2)(x^2 - 4) - x(x^2 - 4) = 0$
(4) $(x^2 - 4)(x^2 - 2 - x) = 0$
(5) $(x + 2)(x - 2)(x + 1)(x - 2) = 0$

The four answers to this equation would then be

$$x = 2;\ x = 2;\ x = -2;\ x = -1.$$

EXERCISE LXXIX.

Solve the equations:

1. $x^3 + 2x^2 = 9x + 18.$
2. $x^3 - 12 = 4x - 3x^2.$
3. $100 - x^3 = 4x^2 - 25x.$
4. $16x + 80 - x^3 = 5x^2.$
5. $180 + x^3 = 5x^2 + 36x.$
6. $x^3 - 27 = 19(x - 3).$
7. $x^3 + 64 = 5x^2 + 22x + 8.$
8. $x^4 - 13x^2 + 36 = 3x(x^2 - 9).$
9. $x^4 - 13x^2 - 48 = 4x^3 - 64x.$
10. $x^4 - 25x^2 = 9x^2 - 225.$

144. In some examples (for instance, in the tenth of the preceding exercise) easier methods of factoring may be seen on simplifying the equation; in others, the simplified form obtained by uniting similar terms is harder to handle.

EXERCISE LXXX.

Solve the following equations:

1. $\dfrac{x^2+3x}{x-2} = 3x+9.$

2. $\dfrac{x^2+x-2}{3} - \dfrac{x^2-x+2}{4} = \dfrac{1}{6}.$

3. $\dfrac{x^2-5x}{2x^2-1} = \dfrac{2}{3}.$

4. $\dfrac{1+x+2x^2}{1-x} = 2+3x.$

5. $\dfrac{3x-1}{x-2} = \dfrac{2x-1}{x-3}.$

6. $\dfrac{1}{x} + \dfrac{1}{x+3} = \dfrac{1}{5}.$

7. $\dfrac{2}{x^2-3} = \dfrac{3}{x-2}.$

8. $\dfrac{x^2-5}{2} = \dfrac{3x-1}{4}.$

9. $\dfrac{1}{x} = \dfrac{13}{x} - 7 + x.$

10. $\dfrac{x}{3} = \dfrac{3}{x-3}.$

11. In the equation $x^2 + xy + 2y^2 = 74$, if $y = 5$ what is the value of x?

12. In the equation $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{14}{45}$, if $x = 5$ what is the value of y?

13. In the equation $4(x + y) = 3xy$, if $x = 4$ what is the value of y?

14. In the equation $x + 2y + \dfrac{3x}{y} = 16$ substitute 5 for x and then find the value of y.

15. In the equation $xy = 2400$ substitute $100 - x$ for y and solve for x.

16. In the equation $\dfrac{1}{x} + \dfrac{1}{y} = 2$ substitute $2 - x$ for y and solve for x.

17. In the equation $35 = 3n + \frac{n(n-1)}{2}d$ substitute 2 for d and solve for n.

18. In the equation $s = an + d\frac{n(n-1)}{2}$ substitute $s = 119$, $a = 2$, $d = 5$, and solve for n.

19. In the equation $s = \frac{n}{2}(a + l)$ substitute $s = 105$, $a = -5$, $l = -5 + (n-1)d$, $d = 9$, and solve for n.

20. In the equation $H = \frac{2Pp}{P + p}$ let $H = 12$, $P = x + 3$, $p = x - 2$, and solve for x.

CHAPTER VI.

THE FIRST METHOD OF ELIMINATION.

145. The equations of condition * already studied have been true only **on condition that** the unknown quantity had a particular value, or one of two or three—one of a few—particular values. Thus the equation

$x + 7 \quad = 2x - 3$ is true only when $x = 10$.
$x^2 + 12 \quad = 7x$ is true only when $x = 3$ or 4.
$(x^2 - 6)^2 = x^2$ is true only when $x = 3$ or -3 or 2 or -2.

But when an equation contains more than one unknown quantity, the condition implied by it becomes much more indefinite. The equation

$$2x - 3y = 6$$

is true for any value of x whatever; but only on condition that when x has any value, y has a particular value which is said to **correspond** to the stated value of x. Suppose, for instance, $x = 5$. Then $2x = 10$ and the equation $2x - 3y = 6$ becomes $10 - 3y = 6$. Then

$$10 = 6 + 3y; \quad 4 = 3y; \quad 1\tfrac{1}{3} = y.$$

If $x = 5$, then the equation is true only on condition that $y = 1\frac{1}{3}$.

If $x = 6$, y must equal 2.

If $x = 7$, $y = 2\frac{2}{3}$; if $x = 8$, $y = 3\frac{1}{3}$; if $x = 9$, $y = 4$.

* See p. 88.

146. There is no end to the solution of this equation; we might continue to write **corresponding values** of x and y as long as we chose, and the solution would still be incomplete.

THE GRAPHICAL METHOD.

147. A method of completely exhibiting the solutions of this kind of equations will now be described. The same method is often used for other purposes in science and in practical affairs.

148. Different values of one letter, like x, may be represented by equally distant points along a straight line; taking a particular point to represent zero, positive whole

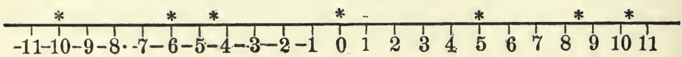


FIG. 1.

numbers are represented by equally distant points running out towards the right, while negative whole numbers are represented by equally distant points running out towards the left.

Fractional numbers of all kinds are represented by points lying between the whole numbers; thus 5 is five spaces to the right of zero, -6 is six spaces to the left of zero, $10\frac{1}{3}$ is ten and one third spaces to the right of zero, $-4\frac{1}{2}$ is four and one half spaces to the left of zero, etc.

On the accompanying diagram the stars mark approximately the points which represent -10, -6, $-4\frac{1}{2}$, 0, 5, $8\frac{1}{2}$, $10\frac{1}{3}$, which may be different values of the letter x.

The only numbers that have no points on this scale to represent them are the imaginary numbers.*

* See § 134.

149. Now we want a method of representing at the same time by one point a value of x and also a value of y. This may be managed by taking two scale-lines like the one just described, and placing them perpendicular to each other with the zero-points together.

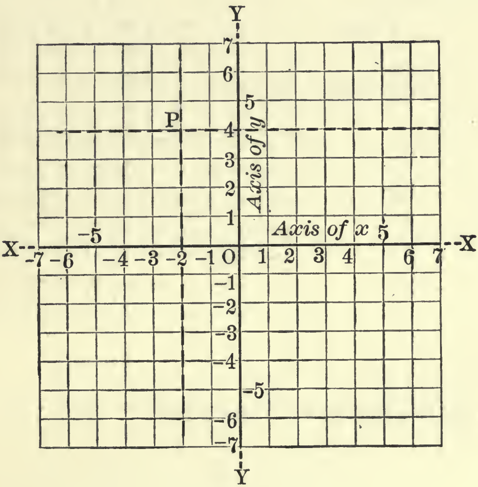


FIG. 2.

Call the horizontal one the x-scale, or axis of x, and the vertical one the y-scale, or axis of y.

Then imagine vertical lines drawn through each point of the axis of x, and horizontal lines through each point of the axis of y.

A point which represents $x = -2$; $y = 4$ must be the point which lies opposite -2 in the axis of x, and opposite 4 in the axis of y. That would be the point marked P in Fig. 2.

150. The points in Fig. 3 that represent the following values of x and y are marked with the corresponding Roman numerals:

I $x = 4;\ y = 2$

II $x = -2;\ y = 3$

III $x = 3;\ y = 5$

IV $x = 6;\ y = -3$

V $\begin{cases} x = 2\frac{1}{2}; \\ y = -2\frac{1}{2} \end{cases}$

VI $\begin{cases} x = -2\frac{2}{3}; \\ y = -3 \end{cases}$

VII $\begin{cases} x = -5; \\ y = -3\frac{2}{3} \end{cases}$

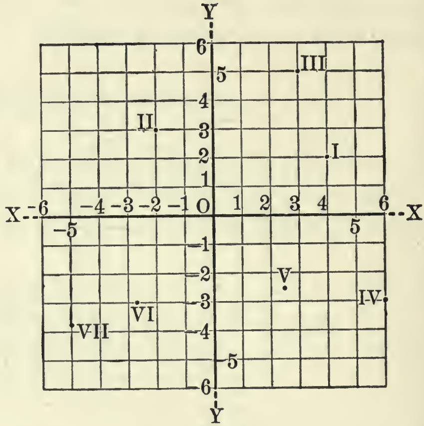


FIG. 3.

151. Returning now to our equation $2x - 3y = 6$, we

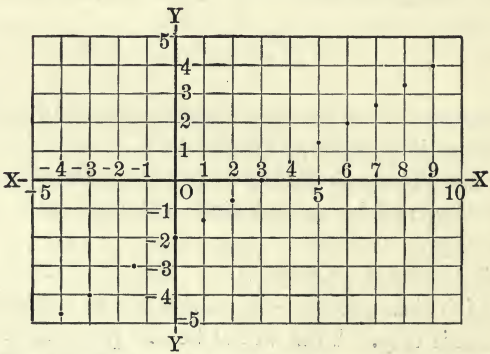


FIG. 4.—Partial list of answers to the equation $2x - 3y = 6$.

find the following set of answers, represented by their respective points, running up from left to right on Fig. 4:

$x = -4;\ y = -4\frac{2}{3}$	$x = 4;\ y = \frac{2}{3}$
$x = -3;\ y = -4$	$x = 5;\ y = 1\frac{1}{3}$
$x = -1\frac{1}{2};\ y = -3$	$x = 6;\ y = 2$
$x = 0;\ y = -2$	$x = 7;\ y = 2\frac{2}{3}$
$x = 1;\ y = -1\frac{1}{3}$	$x = 8;\ y = 3\frac{1}{3}$
$x = 2;\ y = -\frac{2}{3}$	$x = 9;\ y = 4$
$x = 3;\ y = 0$	

An Infinite List of Answers.

152. It would be found on trial that all the other answers to this equation would lie on the same line, i.e., their corresponding points would fall into line with those already put down. In fact, the straight line drawn through two

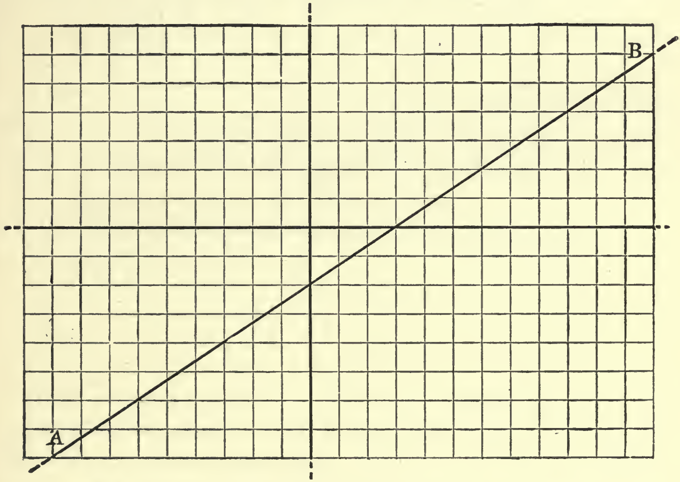


FIG. 5.—Complete list of answers to the equation $2x - 3y = 6$.

points that represent any two answers to the equation is a **complete list** of all possible answers to the equation. In order to be complete, of course, not only the two axes, but also the line AB, are supposed to be endless in extent.

Model A.—In a similar way construct a complete list of answers to the following three equations (see Fig. 6):

$$\begin{array}{ll} \text{I} & 7x + y = 14 \\ \text{II} & 2x + 6y = 3 \\ \text{III} & 2x + 5y = 10 \end{array}$$

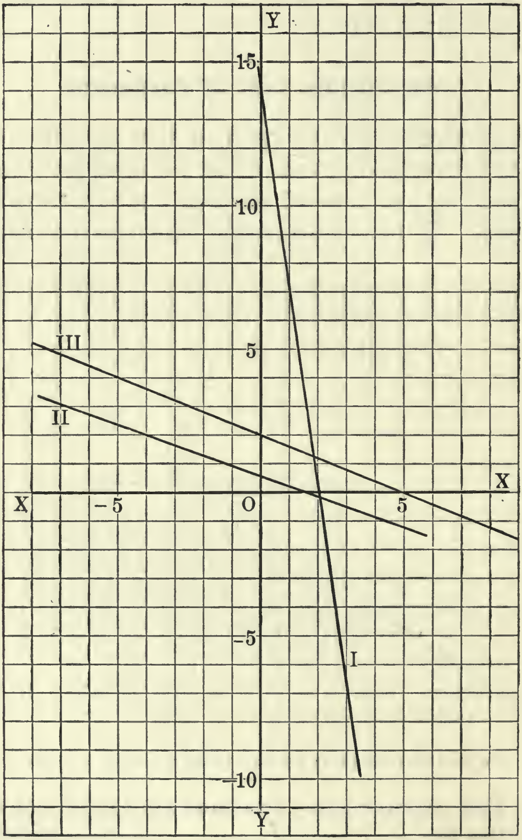


FIG. 6.

[Only two points need be found; then a straight line can be drawn through them.]

EXERCISE LXXXI.

For each of the following equations construct a complete list of answers, and put each pair on a separate sheet: *

1. $x + 2y = 5;\ 3x + 7y = 17.$
2. $15x + 3y = 39;\ 17x - y = 31.$
3. $7x + 4y = 41;\ 3x - y = 4.$
4. $2x + y = 11;\ 3x + 7y = 22.$
5. $5x + 6y = 51;\ 6x - 5y = -12.$
6. $2x - y = -4;\ 7x + 6y = 62.$
7. $8x - y = 27;\ x + 8y = 44.$
8. $15x + 7y = 162;\ 9x + 2y = 84.$
9. $14x - 3y = 44;\ 6x + 17y = 92.$
10. $5x - 7y = 0;\ 7x + 5y = 74.$

153. It will be found that each of these pairs of lines intersect in the points given below; the points are numbered to correspond with the pairs of equations:

1. $x = 1;\ y = 2.$
2. $x = 2;\ y = 3.$
3. $x = 3;\ y = 5.$
4. $x = 5;\ y = 1.$
5. $x = 3;\ y = 6.$
6. $x = 2;\ y = 8.$
7. $x = 4;\ y = 5.$
8. $x = 8;\ y = 6.$
9. $x = 4;\ y = 4.$
10. $x = 7;\ y = 5.$

154. There is only one point, then, that will satisfy two equations of condition of the kind used so far in this chapter. That is, **two conditions serve to determine the values of two unknown quantities.**

155. It must not be rashly assumed that EVERY equation with two unknown letters will have for its complete

* Paper ruled in squares, called cross-section paper, can be bought of dealers in drawing materials.

list of answers a STRAIGHT line; for example, Fig. 7 gives the list of answers for (I) $x^2 = 3y + 10$, and for

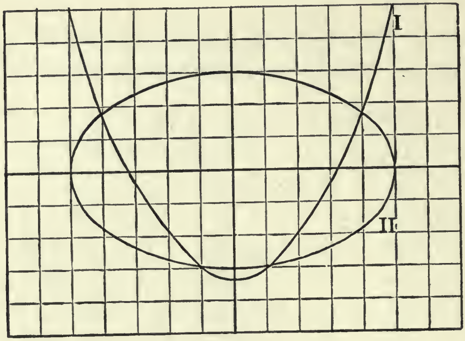


FIG. 7.

(II) $9x^2 + 25y^2 = 225$. There are evidently FOUR different answers that would make BOTH these equations true.

To construct these curves, successive values of y are substituted in each equation, and the values of x calculated by solving the equations so obtained. Thus, in (I), if $y = 2$, $x = 4$ or $x = -4$; if $y = -2$, $x = 2$ or $x = -2$; if $y = 1$, $x = \pm\sqrt{13} = 3.606$ or -3.606; and so on. But if y is taken equal to -4, $x = \sqrt{-2} = 1.414i$ or $-1.414i$, neither of which can be represented by a point on the diagram. In (II), if x is taken greater than 5, or y greater than 3, the results are also imaginary.

Such curves as these form the subject-matter of an extensive and very interesting branch of mathematics, called Analytic Geometry. Not much time can be profitably spent on them here.

SIMULTANEOUS EQUATIONS.

156. Model B.—Six horses and 11 cows sell for $810; 13 horses and 5 cows sell at the same rate for $1190. Price for each animal ?

Let $x =$ the number of dollars for one horse;
$y =$ the number of dollars for one cow.

(I) $6x + 11y = 810$

(II) $13x + 5y = 1190$

Model C.—Six pounds of tea and 11 pounds of coffee sell for $9.23; 13 pounds of tea and 5 pounds of coffee sell at the same rate for $11.90. Price of each ?

Let $x =$ the number of cents for one pound tea;
$y =$ the number of cents for one pound coffee.

(III) $6x + 11y = 923$

(IV) $13x + 5y = 1190$

In the four equations that come from these two problems x and y do not have the same meaning; in (I) and (II) x and y stand for a number of dollars, in (III) and (IV) for a number of cents. When the unknown letters in two equations stand for entirely different quantities, as in (II) and (III), they must be treated as if written with different letters; the terms which seem to be similar are really NOT similar. In (I) and (II) we should find that $x =$ $80 and $y =$ $30; while in (III) and (IV) we should find that $x = 75$ cents and $y = 43$ cents.

157. When in two equations the two letters stand for the same unknown quantities, the equations are called **simultaneous.** Terms having the same letters are similar terms. The members of either equation can be added to or subtracted from the corresponding members of the other equation,* and if the x terms (or the y terms) are the same,

* See § 17.

a resulting equation may be obtained free from x (or y). Thus, for Model B:

$$\begin{array}{lll} \text{(I)} & 6x + 11y = 810 & \\ \text{(II)} & 13x + 5y = 1190 & \\ \hline ③ & 30x + 55y = 4050 & \text{(I)} \times 5 \\ ④ & 143x + 55y = 13090 & \text{(II)} \times 11 \end{array}$$

The y terms are now alike. Subtracting ③ from ④,—

$$\begin{array}{lll} ⑤ & 113x = 9040 & ④ - ③ \\ ⑥ & x = 80 & ⑤ \div 113 \end{array}$$

To find the value of y we may return to the original equations, (I) and (II), make the x terms alike by multiplying (I) by 13 and (II) by 6, and then subtract the resulting equations. Or, since we know now that $x = 80$, $6x = 480$, we may write, instead of (I),

$$\begin{array}{lll} ⑦ & 480 + 11y = 810 & ⑥ \text{ substituted in (I)} \\ ⑧ & 11y = 330 & ⑦ - 480 \\ ⑨ & y = 30 & ⑧ \div 11 \end{array}$$

The answer to Model B, then, is \$80 for each horse, and \$30 for each cow. In the same way we find, for Model C, 75 cents a pound for tea, and 43 cents a pound for coffee.

158. This is the method by which the points of crossing were found for the ten pairs of lines that the pupil was asked to construct. One equation, in any pair, imposes upon x and y the condition that their point must lie in a certain line; the other equation imposes also the condition that the point must lie in another line; and since the point must lie in both lines, it must be the point where the lines cross.

159. That is, taking the first pair of equations in the set of ten mentioned, the two conditions contained in the equations $x + 2y = 5$; $3x + 7y = 17$ are the same as these two conditions: $x = 1$; $y = 2$. The algebraic work consists in reducing the given conditions to the form of particular values for x and y.

Elimination by Combination.

160. The process of getting rid of an unknown letter is called "eliminating" that letter. The method used above is called **Elimination by Combination.** Other methods will be described later.

EXERCISE LXXXII.

Solve the following equations: *

1. $2x + 7y = 59;\ 3x + 4y = 43.$
2. $5x + 8y = 34;\ 9x + 2y = 24.$
3. $13x + 17y = 107;\ 2x + y = 10.$
4. $14x + 9y = 100;\ 7x + 2y = 30.$
5. $x + 15y = 37;\ 3x + 7y = 35.$
6. $15x + 19y = 79;\ 35x + 17y = 157.$
7. $6x + 4y = 90;\ 3x + 15y = 123.$
8. $39x + 27y = 213;\ 52x + 29y = 249.$
9. $72x + 14y = 170;\ 63x + 7y = 112.$
10. $101x + 26y = 886;\ 103x + 39y = 941.$

†11. $2x + 7y = 49;\ 6x - 5y = 17.$

12. $7x - 4y = 2;\ 25x - 13y = 24.$
13. $x + y = 90;\ x - y = 13.$
14. $4x + 9y = 50;\ 7x - 17y = 22.$
15. $5x - 3y = 17;\ 12y - 7x = 23.$
16. $171x - 213y = 642;\ 114x - 326y = 244.$
17. $9y - 7x = 43;\ 15x - 7y = 43.$
18. $12x + 7y = 176;\ 3y - 19x = 3.$
19. $43x + 2y = 266;\ 12x - 17y = 4.$
20. $5x + 9y \quad 188;\ 13x - 2y = 57.$
21. $4x + 3y = 22;\ 3x + 5y = 11.$
22. $5x - 2y = 51;\ 17x - 3y = 162.$

* That is, reduce the given conditions to the simplest form.

† It may happen that the x terms (or the y terms) are of opposite signs in the two equations; in that case the equations must be added to get rid of that letter.

23. $3x - 5y = 51;\ 5x + 7y = 39.$

24. $7y - 5x = 143;\ 3x + 5y = 89.$

25. $4x + 11y = 144;\ 8x + 17y = 198.$

26. $2x - 7y = 8;\ 3y - 9x = 21.$

27. $17x + 12y = 59;\ 19x - 3y = 148.$

28. $8x + 3y = 3;\ 4x + 3y = 1.$

29. $59y - 17x = 123;\ 2x - 13y = -17.$

30. $3x + 2y = 42;\ 13x + 23y = 225.$

31. $\begin{cases} 80x + 50y = 22190; \\ 70x = \frac{28}{25}(60y). \end{cases}$

32. $\begin{cases} \dfrac{2x}{3} + \dfrac{3y}{4} = 700; \\ y + 100 = \dfrac{5x}{6}. \end{cases}$

33. $\begin{cases} x + y = 7; \\ \dfrac{5 + x}{9 + y} = \dfrac{3}{4}. \end{cases}$

34. $\begin{cases} x + y = \dfrac{15}{16}. \\ x - y = \dfrac{13}{16}. \end{cases}$

35. $\begin{cases} \dfrac{2x}{3} + \dfrac{y}{2} = 60; \\ \dfrac{4x}{5} - y = 8. \end{cases}$

36. $\begin{cases} x + 60 = \dfrac{4y}{5}; \\ x + y = 120. \end{cases}$

37. $\begin{cases} \dfrac{x + 2}{y} = \dfrac{1}{2}; \\ \dfrac{x}{y + 3} = \dfrac{1}{3}. \end{cases}$

38. $\dfrac{x + 3}{y + 3} = \dfrac{5}{6};\quad \dfrac{x - 3}{y - 3} = \dfrac{2}{3}.$

39. $x - y = 5;\quad \dfrac{12 - x}{17 - y} = \dfrac{2}{7}.$

40. $\begin{cases} x + y = 7; \\ 10x + y + 9 = 10y + x. \end{cases}$

EXERCISE LXXXIII.

1. For 8 cows I get \$20 more than I pay for 40 sheep; and for 16 sheep I pay \$21 more than I get for 3 cows. Price of each?

2. Find two numbers whose difference is $\frac{1}{35}$ of their sum, and 3 less than $\frac{1}{9}$ of the larger number.

3. In 10 hours A walks 1 mile less than B does in 8

hours; and in 6 hours B walks 1 mile less than A does in 8 hours. How many miles does each walk per hour?

4. If A's money were increased by 36 cents he would have 3 times as much as B; if B's money were increased by 5 cents he would have half as much as A. How much has each?

5. A pound of tea and 6 pounds of sugar cost 72 cents; if sugar were to rise 50 per cent and tea 10 per cent, the same quantity would cost 84 cents. Price of tea and of sugar?

6. Find two numbers such that 3 times the greater exceeds twice the less by 29, and twice the greater exceeds 3 times the less by 1.

7. Three men and 16 boys earn \$107.25 in $5\frac{1}{2}$ days, and 4 men with 10 boys earn \$192.50 in 11 days. What are the daily wages of men and of boys?

8. A farmer bought land for \$7500, part at \$80 per acre and part at \$50 per acre. The cheaper he sold at a loss of 10 per cent, and the dearer at a gain of 10 per cent. On the whole he profited \$50 by the transaction. How much land did he buy?

9. If I divide the smaller of two numbers by the greater, the quotient is .36 and the remainder .64; if the greater is divided by the smaller, the quotient is 2 and the remainder 27. Find the numbers.

10. The sum of the ages of a father and son will be doubled in 25 years; the difference of their ages is $\frac{1}{3}$ of what the sum will be in 20 years. How old are they?

11. Find two numbers such that 3 times the greater exceeds $\frac{1}{3}$ the less by 439, and 3 times the less exceeds $\frac{1}{3}$ the greater by 49.

12. Eight years ago A was 5 times as old as B; 7 years hence he will be only twice as old. How old are they now?

13. A merchant offers me 8 pounds of black tea and 24

pounds of green tea for \$10, or 14 pounds of black tea and 17 pounds of green tea for the same sum. If it makes no difference to him which offer I accept, what are his prices per pound?

14. A and B buy a horse for \$550; A can pay for it if B will advance $\frac{1}{3}$ the money he has in his pocket; and B can pay for it if A will advance $\frac{1}{4}$ the money in his pocket. How much has each?

15. A sum of money was divided equally among a certain number of persons; if there had been 10 more persons, each would have received \$2 less; if there had been 10 less, each would have received \$3 more. How many persons, and the share of each?

16. If a certain lot of land were 8 feet wider and 2 feet longer, it would contain 960 square feet more; if it were 2 feet narrower and 8 feet shorter, it would contain 760 square feet less. What is its area?

17. A farmer bought eggs at 2 for 5 cents, and others at 3 for 8 cents; wishing to sell them to a relative at cost, he named a price of 31 cents per dozen, but found he would lose 5 cents; his relative then suggested 5 for 13 cents, and the farmer accepted that, but made a profit of 10 cents on the transaction. How many dozen were there of each kind?

18. It takes me 16 times as long to walk around the edge of a long rectangular field as to walk directly across it; and the next field, which is 30 feet wider and 150 feet shorter, has the same area. What are the dimensions of each field?

19. A cruiser 386.8 feet long passes a battleship going in the same direction in 1.12 minutes; returning, she passes the same battleship in 12 seconds. The length of the battleship is 352.4 feet. Supposing each ship maintains the same speed throughout, how many miles per hour do they go (5280 feet to the mile)?

20. Two men who lived in towns 70 miles apart started at the same time and rode towards each other; when they met, they exchanged horses and continued as before. One horse could go 10 miles per hour, the other only 8. The slower horse arrived while the faster horse was still 5 miles from the journey's end. Find the time each traveller took on the road.

Find the fractions described as follows:

21. If 4 be added to the numerator, the value of the fraction will become 1;
If 3 be added to the denominator, the value of the fraction will become $\frac{1}{2}$.

22. If 1 be subtracted from the numerator, the value of the fraction will become $\frac{1}{2}$;
If 11 be added to the denominator, the value of the fraction will become $\frac{1}{3}$.

23. If 8 be added to the numerator, the value of the fraction will become $\frac{3}{4}$;
If 8 be added to the denominator, the value of the fraction will become $\frac{1}{4}$.

24. If 17 be added to numerator and to denominator, the value of the fraction will become $\frac{5}{6}$;
If 1 be subtracted from numerator and from denominator, the value of the fraction will become $\frac{1}{3}$.

25. If 3 be taken from the numerator and 1 be added to the denominator, the value of the fraction will become $\frac{1}{4}$;
If 3 be added to the numerator and 5 to the denominator, the value of the fraction will become $\frac{1}{2}$.

26. Find two fractions with numerators 4 and 11 respectively, such that their sum is $2\frac{13}{35}$, and if their denominators are interchanged their sum is $2\frac{27}{35}$.

27. A fraction is such that when it is multiplied by $\frac{2}{3}$ the sum of its terms is 107; if 8 is added to each of its terms the value of the fraction becomes $\frac{3}{5}$.

28. A proper fraction is such that if the numerator is halved and the denominator increased by 3 its value becomes $\frac{1}{5}$; if the fraction is multiplied by $\frac{4}{5}$ the difference of its terms will be 53.

29. The sum of the terms of a fraction, divided by their difference, gives 2 for a quotient and 5 for a remainder. If the fraction were divided by 2 the sum of its terms would be 53.

30. Two fractions have denominators 20 and 10 respectively. The fraction formed from these two by taking for its respective terms the sums of the corresponding terms of these two fractions is $\frac{2}{3}$; and the fraction similarly formed by taking the differences is $\frac{3}{5}$.

Find the numbers described as follows:

31. A number divided by the sum of its digits gives $7\frac{3}{4}$ for a quotient; if 54 be subtracted from the number, the digits are reversed.

32. If I divide a number by the sum of its digits, the quotient is 6 and the remainder 7; but if I invert the order of the digits and then divide by the sum of the digits, the quotient is 4 and the remainder 6.

33. A number exceeds 5 times the sum of its digits by 8; if the order of the digits were reversed, the number would be 15 less than 7 times the sum of the digits.

34. A number is 4 times the sum of its digits; and if 27 be added to the number, the order of the digits is reversed.

35. A number is 2 more than 5 times the difference of its digits; if its digits are reversed, the resulting number is 8 times the sum of its digits.

36. Two numbers have the same digits in opposite order; the difference of the numbers is 3 times the sum of the digits; and the sum of the digits is 2 more than $2\frac{1}{2}$ times the difference of the digits.

37. If 45 be subtracted from a number, the digits are

reversed; but the same result might have been obtained by subtracting 7 from the number and then dividing by 2.

38. Two numbers which have the same digits in opposite order differ by 18, and the smaller number is 4 times the sum of the digits.

39. Subtracting 27 from a number reverses the digits; in the scale of 8, instead of the decimal scale, the number would be 18 less.

40. If a certain number is divided by the sum of its digits, the quotient is 4 and the remainder 6; if the digits are reversed, the resulting number divided by 5 gives 4 more than the sum of the digits.

41. A number consists of three digits, the middle digit being zero. If the digits are reversed, the number is increased by 396; if only the second and third digits change places, the number is increased by 3 more than 6 times the sum of its digits.

42. A number of three digits, the middle digit being 6, has its digits reversed if one adds to it 22 times the sum of its digits; and the number so formed is 72 less than double the original number.

43. A number has four digits, of which the second is 6 and the fourth is 3; reversing the digits increases the number by 909; but if only the first and third digits change places, the number is increased by 2970.

Solving for Reciprocals.

161. In equations like the following it is better to make the x terms (or the y terms) alike WITHOUT first clearing of fractions:

Model D.

①	$\dfrac{3}{x}+\dfrac{2}{y}=\dfrac{2}{7}$	
②	$\dfrac{5}{x}+\dfrac{4}{y}=\dfrac{11}{21}$	
③	$\dfrac{6}{x}+\dfrac{4}{y}=\dfrac{4}{7}$	① × 2
④	$\dfrac{1}{x}=\dfrac{1}{21}$	③ − ②
⑤	$21=x$	④ × $21x$

and similarly for y.

EXERCISE LXXXIV.

1. A and B together can do a piece of work in $8\frac{2}{5}$ days; if A worked 3 days and B 5 days, only half of the work would be done. How long for A alone? How long for B?

2. Five boys and 10 men could do a certain job in 3 days; one man and one boy would take 24 days to do it. How long for one man alone to do the work? One boy alone?

3. Twenty-four pails of water and 20 cans of milk will just fill a certain tank; 6 pails and 14 cans will half fill it. How many pails to fill it? How many cans?

4. Two pipes fill a cistern in 20 minutes; if one of the pipes were twice as large * and the other half as large, the cistern would be filled in 15 minutes. How long for each pipe?

* The word "large" here refers not to the width but to the capacity of the pipe.

5. Two pipes can fill a cistern in 5 minutes; if one of the pipes is closed half the time it will take $7\frac{1}{2}$ minutes. How long for each pipe alone?

6. A and B could dig a well in 12 days; but at the end of the third day B quits, so that the job lasts A 6 days longer. How long for each alone?

7. After working 2 days on a certain job with B, A says to him, "I can finish this job alone in 10 days." B replies, "If we work together one more day, I can finish it alone in 5 days." If what they say is true, how long will it take each alone to finish the job?

8. A can row 11 miles down-stream for every 7 against the stream; he rows down-stream for 3 hours, then rows back, and at the end of the 6 hours he is 5 miles from his starting-place. How fast does he row, and how fast does the stream flow?

9. B rows 9 miles down-stream in 45 minutes; he rows back, near the bank, where the current is only half-strength, in one hour and a half. Speed of boat and of stream?

10. C rows 6 hours down-stream and 15 hours up-stream, covering in all 72 miles. His speed up-stream is $\frac{2}{3}$ of his speed in still water. Speed of stream?

11. D rows down-stream for an hour and a half, but it takes him 3 hours to row back. He rows in the first 3 hours 12 miles. Speed of boat and of stream?

12. A and B run a mile race; A gives B 12 seconds start and beats him by 44 yards; then A gives B 165 yards start, and is beaten by 10 seconds. Speed of each?

13. A traveller started on a journey of 330 miles, having 4 hours and 12 minutes to spare on a connection he expected to make at the end of that journey; but an accident occurred when he was 2 hours out, which not only held up the train for 2 hours, but diminished its speed for the rest of the run. Then he missed his connection by just 3 min-

utes. If the accident had happened 6 miles further on, he would have been barely in time. How fast did the train go before and after the accident?

14. A was sent to a town 147 miles away, and 7 hours later B was sent after him. After travelling 71 miles B was handed a letter to deliver to a person living 17 miles out of the town to which A had been sent. He overtook A just as he was entering the town, and handed him the letter; the letter was delivered just 9 hours and 40 minutes after B received it. Speed of A and of B?

Where Elimination Fails.

162. In studying equations of one unknown letter, we found that the given equation could be reduced to a very simple equation giving a particular value for that letter. That simple equation was the condition that the given equation should be true.

We afterwards found that if there were two unknown letters in an equation of condition, another equation had to be given before we could get particular values of x and y. When only one equation is given, either of the two letters can have any numerical value; there must be given two equations of condition before their values are limited to a particular set.

163. In this connection the following problem is instructive:

Model E.—A certain number of two digits is equal to 4 times the sum of its digits; if the digits are reversed, it is equal to 21 times the difference of its digits.

Let $x =$ the tens digit;
$y =$ the units digit.

$$\begin{array}{ll} (1)\ 10x + y = 4(x + y) & \\ (2)\ 10y + x = 21(x - y) & \\ \hline (3)\ 10x + y = 4x + 4y & \text{same as } (1) \end{array}$$

④ $6x - 3y = 0$	③ $- 4x - 4y$
⑤ $2x - y = 0$	④ $\div 3$
⑥ $10y + x = 21x - 21y$	same as ②
⑦ $31y - 20x = 0$	⑥ $- 21x + 21y$

If we compare equations ⑦ and ⑤ we shall see that both cannot at the same time be true; ⑤ says that $x = \frac{1}{2}y$, while ⑦ says that $x = \frac{31}{20}y$.

164. Such equations are called **inconsistent**, and of course cannot be used to determine the values of the unknown quantities. They point to some mistake in the statement of the problem, or in the pupil's understanding of it.

165. We tacitly took for granted, in forming equation ②, that the tens digit was larger than the units. Let us see if that is the source of the inconsistency.

① $10x + y = 4x + 4y$
② $10y + x = 21y - 21x$

. . . .

⑥ $22x - 11y = 0$	② $+ 21x - 21y$
⑦ $2x - y = 0$	⑥ $\div 11$

Now the equations ① and ② are consistent,—too much so in fact; for the second equation reduces to one precisely like the first. In this case also, then, we cannot determine, from the two equations given, the particular values of x and y.

166. In all equations of this kind, then, we must require that they be **consistent**, that is, that both CAN be true at the same time; and that they be **independent**, that is, that both must not reduce to the same equation.

Non-Algebraic Conditions.

167. It happens sometimes that a problem is given, like the above, which implies only one ALGEBRAIC condition, but still the answer is restricted to a few or perhaps even

to only one particular value, by a condition, implied in the statement of the problem, such that it cannot be stated algebraically.

In Model E our digits must be from the nine Arabic numerals; we found by trial that our equations were inconsistent and the problem unsolvable unless the second digit were the larger; and our algebraic conditions both reduced to the fact that the second digit was twice the first. That leaves, for the only possible solutions,—

$x = 1$; $y = 2$; the number 12
$x = 2$; $y = 4$; the number 24
$x = 3$; $y = 6$; the number 36
$x = 4$; $y = 8$; the number 48

Four particular answers to the problem.

168. Equations which do not give a solution from the algebraic conditions alone are called **indeterminate equations**; and the problems which give rise to such equations, although they may contain enough non-algebraic conditions to determine the answer, are often called indeterminate problems.

More Than Two Unknown Letters.

169. The equation $3x + 4y + 6z = 41$ has three unknown letters in it. If we take the value of x to be 3, the equation reduces to $2y + 3z = 16$; and if we take the equation $x + 2y + 3z = 19$ and eliminate x thus:

$$\begin{array}{lll} (1) & 3x + 4y + 6z = 41 & \\ (2) & x + 2y + 3z = 19 & \\ (3) & 3x + 6y + 9z = 57 & (2) \times 3 \\ (4) & 2y + 3z = 16 & (3) - (1) \end{array}$$

we come to the same result.

In this case we have imposed the same condition in two different ways; and from our experience with equations

like $2y + 3z = 16$ we know that the values of y and z cannot be determined unless we have one more condition.

Counting the given equation of condition, the second that we imposed at random (since $x = 3$ and $x + 2y + 3z = 19$ are not independent conditions we have only imposed one), and the third that we know we must have, there are **three conditions necessary to determine three unknown values.**

If the values turn out to be

$$x = 3; \quad y = 5; \quad z = 2,$$

or whatever else they are, these answers would themselves be THREE independent equations; and to get them we must have three independent equations to start with.

170. Whatever the number of unknown letters, the same number of independent equations of condition is required to determine their values.

Elimination with Three Letters.

171. Model F.

$$\left.\begin{array}{lr} (1) & 3x + 4y + 5z = 39 \\ (2) & x + 2y + 3z = 19 \\ (3) & 4x + 5y + 7z = 51 \end{array}\right\} \textit{Three given conditions.}$$

$$\begin{array}{lrl} (4) & 3x + 6y + 9z = 57 & (2) \times 3 \\ (5) & 2y + 4z = 18 & (4) - (1) \\ (6) & y + 2z = 9 & (5) \div 2 \end{array}$$

We have now eliminated x from the first two equations. We might have eliminated y or z from the same two equations, or we might have taken (1) and (3) or (2) and (3), instead of the first pair.

Now that we have started with x, however, we must continue to eliminate this letter; (6) is an equation with y and z for letters, and we need another equation to determine their values; so we must eliminate x again; and we must take a different pair of equations, either (1) and (3) or (2)

and (3), or else our second equation in y and z will not be independent of (6).

$$(7)\quad 4x + 8y + 12z = 76 \qquad (2) \times 4$$
$$(8)\quad 3y + 5z = 25 \qquad (7) - (3)$$

(6) and (8) are two independent equations with two unknown letters; and the rest of the solution is familiar ground.

$$(9)\quad 3y + 6z = 27 \qquad (6) \times 3$$
$$(10)\quad z = 2 \qquad (9) - (8)$$
$$(11)\quad y + 4 = 9 \qquad (10) \text{ substituted in } (6)$$
$$(12)\quad y = 5 \qquad (11) - 4$$
$$(13)\quad x + 10 + 6 = 19 \qquad (10) \text{ and } (12) \text{ subst. in } (2)$$
$$(14)\quad x = 3 \qquad (13) - 16$$

Ans. $x = 3$; $y = 5$; $z = 2$.

172. When we have obtained two equations from which one of the unknown quantities is eliminated, these are called the equations of the **new set**. In the preceding solution equations (6) and (8) are the equations of the new set.

EXERCISE LXXXV.

1. $\begin{cases} 5x + 7y - 2z = 13. \\ 8x + 3y + z = 17. \\ x - 4y + 10z = 23. \end{cases}$

2. $\begin{cases} 3x - 2y + 5z = 16. \\ 3x - 2y + 4z = -10. \\ 4x + 2y - 5z = -2. \end{cases}$

3. $\begin{cases} 5x + 3y - 6z = 4. \\ 3x - y + 2z = 8. \\ x - 2y + 2z = 2. \end{cases}$

4. $\begin{cases} 2x - 2y + z = 1. \\ 2x + 3y - z = 5. \\ x + y + z = 6. \end{cases}$

5. $\begin{cases} 2x + 3y + 4z = 20. \\ 3x + 4y + 5z = 26. \\ 3x + 5y + 6z = 31. \end{cases}$

6. $\begin{cases} y - x + z = -5. \\ z - y - x = -25. \\ x + y + z = 35. \end{cases}$

7. $\begin{cases} 3x - 2y + 5z = 26. \\ x - 2y + 3z = 6. \\ 2x + 3y - 4z = 20. \end{cases}$

8. $\begin{cases} 4x - 3y + 2z = 40. \\ 5x + 9y - 7z = 47. \\ 9x + 8y - 3z = 97. \end{cases}$

9. $\begin{cases} 3x + 2y + z = 23. \\ 5x + 2y + 4z = 46. \\ 10x + 5y + 4z = 75. \end{cases}$

10. $\begin{cases} 5x - 6y + 4z = 15. \\ 7x + 4y - 3z = 19. \\ 2x + y + 6z = 46. \end{cases}$

Solve for reciprocals first: *

11. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=36. \\ \dfrac{1}{x}+\dfrac{3}{y}-\dfrac{1}{z}=28. \\ \dfrac{1}{x}+\dfrac{1}{3y}+\dfrac{1}{2z}=20. \end{cases}$

12. $\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}-\dfrac{4}{z}=8. \\ \dfrac{3}{x}-\dfrac{4}{y}+\dfrac{5}{z}=26. \\ \dfrac{5}{x}-\dfrac{3}{y}+\dfrac{2}{z}=23. \end{cases}$

13. $\begin{cases} \dfrac{2}{x}+\dfrac{2}{y}-\dfrac{2}{z}=\dfrac{7}{60}. \\ \dfrac{3}{x}+\dfrac{4}{y}-\dfrac{2}{z}=\dfrac{7}{30}. \\ \dfrac{5}{x}+\dfrac{3}{y}-\dfrac{2}{z}=\dfrac{3}{10}. \end{cases}$

14. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=36. \\ \dfrac{1}{x}+\dfrac{3}{y}-\dfrac{1}{z}=28. \\ \dfrac{1}{x}+\dfrac{1}{3y}+\dfrac{1}{2z}=20. \end{cases}$

15. $\begin{cases} \dfrac{3}{x}-\dfrac{4}{5y}+\dfrac{1}{z}=\dfrac{38}{5}. \\ \dfrac{1}{3x}+\dfrac{1}{2y}+\dfrac{2}{z}=\dfrac{83}{12}. \\ \dfrac{4}{5x}-\dfrac{1}{2y}+\dfrac{4}{z}=\dfrac{121}{20}. \end{cases}$

173. Model G.—From the four equal expressions in the **continued equation**

$\frac{1}{2}(x+z-5)=y-z=2x-11=9-(x+2z)$

we could form six different equations, but only three of them would be independent; from these three the values of x, y, and z can be determined.

① $\frac{1}{2}(x+z-5)=y-z$	
② $y-z=2x-11$	
③ $2x-11=9-(x+2z)$	
④ $x+z-5=2y-2z$	① $\times 2$
⑤ $x-2y+3z=5$	④ $-2y+2z+5$
⑥ $2x-y+z=11$	② $-y+z+11$
⑦ $3x+2z=20$	③ $+x-2z+11$

* See § 161.

Here are three equations, (5), (6), and (7), for three unknown letters; but one of the equations has only two unknown letters in it. The letter missing from that is y, so we combine (5) and (6) to get rid of y, and thus get TWO equations with two unknown letters [(7) and (9)].

$$\begin{array}{lll} (8) & 4x - 2y + 2z = 22 & (6) \times 2 \\ (9) & 3x - z = 17 & (8) - (5) \\ \hline (7) & 3x + 2z = 20 & \\ (9) & 3x - z = 17 & \end{array} \Bigg\} \textit{ New set}$$

Whence we obtain $x = 6$; $y = 2$; $z = 1$. *Ans.*

EXERCISE LXXXVI.

1. $x + 20 = \dfrac{3y}{2} + 10 = 2z + 5 = 110 - (y + z)$.

2. $\dfrac{y+z}{4} = \dfrac{z+x}{3} = \dfrac{x+y}{2}$; $x + y + z = 27$.

3. $\dfrac{y-z}{3} = \dfrac{y-x}{2} = 5z - 4x$; $y + z = 2x + 1$.

4. $x + 2y = 5z - 10x = y + z = 600$.

5. $x - \dfrac{y}{5} = 6$; $y - \dfrac{z}{7} = 8$; $z - \dfrac{x}{2} = 10$.

6. $\dfrac{2}{x} + \dfrac{1}{y} = \dfrac{3}{2z}$; $\dfrac{3}{z} - \dfrac{2}{y} = 2$; $\dfrac{1}{x} + \dfrac{1}{z} = \dfrac{4}{3}$.

7. $2x - 3y = 8$; $y - 3z = -11$; $x - 2y + 4z = 17$.

8. $\dfrac{1}{x} + \dfrac{2}{y} = 5$; $\dfrac{3}{y} - \dfrac{4}{z} = -6$; $\dfrac{3}{z} - \dfrac{4}{x} = 5$.

9. $\dfrac{2x-y}{3} = \dfrac{3y+2z}{4} = \dfrac{x-y-z}{5} = 4$.

10. $\dfrac{x-y}{3} = \dfrac{y-z}{4} = \dfrac{x+z}{5} = \dfrac{x+10}{12}$.

EXERCISE LXXXVII.

1. A storekeeper exchanged with a neighbor 2 bushels of oats and rye, mixed half and half, for $1\frac{2}{3}$ bushels of corn; and he told the neighbor that of two bills that he sent out for corn, oats, and rye, the first was for 5, 6, and 8 bushels respectively, and amounted to \$10.30; the second was for 3, 5, and 8 bushels respectively, and amounted to \$8.75. Price of each kind of grain?

2. Suppose you and I together had 55 cents; you and some friend of ours had 62 cents; while the three of us had 97 cents. How much would each of us have?

3. A pays to B and C as much as each of them has; then B pays to A and C half as much as each of them has after the first division; finally C pays to A and C one-third as much as each of them has after the second division. Counting their shares then, A has \$12, B \$84, and C \$138. How much had each at first?

4. A and B together can do a job in 12 days. After they get it three-quarters done, however, they call on C to help them, and thus save one day. C can do as much work in 5 days as A can in 6. How long for each alone to do the entire job?

5. A and B together can do a certain job in 5 days; if A works 2 days, B 3 days, and C 5 days, $\frac{7}{15}$ of the job will be done; but if A works 6 days and B 3 days, the job will be half done. How long will it take each alone to do the entire job?

6. A number consists of six digits, of which the first is 1; if the first digit is erased and written down after the other five, the result is 3 times the original number. Find the number.

7. A number consists of three digits; their sum is 13, and the middle digit is $\frac{4}{9}$ of the other two; if 297 be added to the number, the digits are reversed.

8. If 5 kegs, 3 cans, and 2 jars of oil be drawn from a full cask, the cask will remain $\frac{11}{30}$ full; if 4 kegs, 5 cans, and 8 jars be drawn from the full cask, the cask will remain $\frac{3}{20}$ full; and if 2 jars be filled from a full keg, the keg will then contain $\frac{1}{60}$ of the cask full. What fraction of the cask full will the keg, the can, and the jar respectively contain?

9. A farmer received \$5745 for horses, cows, and sheep, at prices of \$110, \$62.50, and \$7.50 apiece respectively; 8 more sheep would have brought as much as 6 cows; and the total number of animals was 5 times the number of cows. Find the number of each.

10. A, B, and C subscribed \$100; if C had put in \$2 more, and B $\frac{1}{10}$ more than he did, A could have completed the sum by subscribing $\frac{1}{10}$ less; if C had put in \$18.50 and B $\frac{1}{8}$ more than he did, A could have put in $\frac{1}{8}$ less. Sum each subscribed?

More than Three Letters.

174. Equations of four or more unknown letters are treated in a similar way. From the original set of equations one of the original set of unknown letters is eliminated, and a new set of equations is formed, with

one equation less;
one unknown letter less.

This is repeated until we have one equation and one unknown letter; and then the work is done.

175. Care must be taken that **each equation of a new set is derived from a different pair of the old equations,** so that they may all be independent. The following example shows an error that is often made.

Model H.

$$\left.\begin{array}{ll}
① & 2x + 3y - 2z + 5u = 39 \\
② & 3x + 5y + 4z - 3u = 28 \\
③ & 5x - 2y - 3z + 2u = 32 \\
④ & 4x + 4y - 5z - 4u = 1
\end{array}\right\} \textit{Original set.}$$

$$\begin{array}{lll}
 & ⑤ \ 4x + 6y - 4z + 10u = 78 & ① \times 2 \\
* & ⑥ \ 2y + z + 14u = 77 & ⑤ - ④ \\
 & ⑦ \ 6x + 9y - 6z + 15u = 117 & ① \times 3 \\
 & ⑧ \ 6x + 10y + 8z - 6u = 56 & ② \times 2 \\
* & ⑨ \ - y - 14z + 21u = 61 & ⑦ - ⑧ \\
 & ⑩ \ 12x + 20y + 16z - 12u = 112 & ② \times 4 \\
 & ⑪ \ 12x + 12y - 15z - 12u = 3 & ④ \times 3 \\
* & ⑫ \ 8y + 31z = 109 & ⑩ - ⑪
\end{array}$$

$$\left.\begin{array}{ll}
⑥ & 2y + z + 14u = 77 \\
⑨ & - y - 14z + 21u = 61 \\
*⑫ & 8y + 31z = 109
\end{array}\right\} \textit{New set.}$$

$$\begin{array}{lll}
 & ⑬ \ 6y + 3z + 42u = 231 & ⑥ \times 3 \\
 & ⑭ \ - 2y - 28z + 42u = 122 & ⑨ \times 2 \\
* & ⑮ \ 8y + 31z = 109 & ⑬ - ⑭
\end{array}$$

Since ⑫ and ⑮ are identical, the three equations ⑥, ⑨, and ⑫ are NOT INDEPENDENT. Scanning the work we find that ⑥ was obtained by combining ④ and ①, ⑨ from ① and ②, and ⑫ from ④ and ②; equation ⑫ therefore contains no facts that are not already in ⑥ and ⑨. And if these three equations were independent, we should have determined our set of four unknown quantities from three equations,—which is impossible.

176. In eliminating with several unknown letters, then, one must be careful that for any number of equations of the NEW SET at least **one more** of the equations of the original set has been utilized.

In the work given above, any one of the pairs ④ ①, ① ②, ④ ②, may be replaced by any pair containing ③.

Equation ⑫, obtained from ④②, has the advantage that it can be used without change as one of the next new set; it would be better, then, to put some other pair instead of ④① or ①②. A correct solution of the problem would be as follows:

Model H.

$$\left.\begin{array}{ll} (1) & 2x + 3y - 2z + 5u = 39 \\ (2) & 3x + 5y + 4z - 3u = 28 \\ (3) & 5x - 2y - 3z + 2u = 32 \\ (4) & 4x + 4y - 5z - 4u = 1 \end{array}\right\} \textit{Original set.}$$

	⑤ $4x + 6y - 4z + 10u = 78$	① × 2
*	⑥ $2y + z + 14u = 77$	⑤ − ④
	⑦ $10x + 15y - 10z + 25u = 195$	① × 5
	⑧ $10x - 4y - 6z + 4u = 64$	③ × 2
*	⑨ $19y - 4z + 21u = 131$	⑦ − ⑧
	⑩ $12x + 20y + 16z - 12u = 112$	② × 4
	⑪ $12x + 12y - 15z - 12u = 3$	④ × 3
*	⑫ $8y + 31z = 109$	⑩ − ⑪

$$\left.\begin{array}{ll} (6) & 2y + z + 14u = 77 \\ (9) & 19y - 4z + 21u = 131 \\ {*}(12) & 8y + 31z = 109 \end{array}\right\} \textit{First new set.}$$

	⑬ $6y + 3z + 42u = 231$	⑥ × 3
	⑭ $38y - 8z + 42u = 262$	⑨ × 2
*	⑮ $32y - 11z = 31$	⑭ − ⑬

$$\left.\begin{array}{ll} (12) & 8y + 31z = 109 \\ (15) & 32y - 11z = 31 \end{array}\right\} \textit{Second new set.}$$

⑯ $32y + 124z = 436$	⑫ × 4
⑰ $135z = 405$	⑯ − ⑮
⑱ $z = 3$	⑰ ÷ 135

N.B. It is convenient, as fast as the equations of the new set are obtained, to mark them with an asterisk and thus save perplexity in looking for them.

(19) $8y + 93 = 109$	subst. (18) in (12)
(20) $y = 2$	$[(19) - 93] \div 8$
(21) $4 + 3 + 14u = 77$	subst. (18) and (20) in (6)
(22) $u = 5$	$[(21) - 7] \div 14$
(23) $2x + 6 - 6 + 25 = 39$	subst. (18), (20) and (22) in (1)
(24) $x = 7$	$[(23) - 25] \div 2$

$x = 7;\ y = 2;\ z = 3;\ u = 5.$ *Ans.*

EXERCISE LXXXVIII.

1. $3x + 2y + z + 5w = 30$
$2x - 3y - 5z + w = -15$
$x + 5y - 3z - 2w = -6$
$5x - y - 2z + 3w = 9$

2. $7x - 5y + 10z - 2w = 29$
$3x + 7y - 5z + 10w = 57$
$2x + 3y + 7z - 5w = 16$
$10x - 2y + 3z + 7w = 61$

3. $2x + 5y = 11 + 3z + 2w$
$$x - 7 = \frac{5z - 2y - 3w}{8}$$
$$y - 11 = \frac{2z - 3x - 5w}{8}$$
$x = 1 + \frac{1}{5}(3y + 2w - 8z)$

4. $3x - 2z = 2$
$5y - 7w = 2$
$2y + 3z = 45$
$4z + 3w = 56$

5. $2x + 11z = 61$
$3u - 14x = -12$
$10y + 3x = 59$
$7u + 13z = 83$

6. $8x - 3z + 4u = 23$
$5y - 3z + 2w = 12$
$6y - 4x - 3u = 12$
$5y - 4u + 3w = 11$
$4z - 9w = 30$

7. $2x - 3y + 4z - 15 = 2y - 3z + 4u - 7 = \frac{1}{20}(2z - 3u + 4x)$
$= \frac{1}{10}(x + y + z + u) = 2$

8. $3x - 5z - 2y + 6 = 10y + 2z - 3w + 4 = 5x + 3z - 8w + 4$
$= 13x - 25y - 11w + 5 = 2z$

9. There is a number of four digits such that if the digits are reversed, the number is diminished by 1089; if the second and third digits are interchanged, the number is diminished by 90; if the first two digits are removed and placed after the last two, the number is increased by 693; and if the number were in the scale of 9 instead of in the

decimal scale, its value would be decreased 912. Find the number.

10. A traveller has a large number of foreign bills, all of the same value, also gold, silver, and copper coins, all coins of the same metal having the same value, and is vainly attempting to pay his railroad fare with them; he offers 5 gold coins, 6 silver coins, and 2 bills, but is \$1.10 short; then he tries 6 bills, 2 gold coins, 7 silver coins, and 6 copper coins, and finds that makes \$4.03 too much; 6 gold coins, 8 silver, 10 copper, and one bill is again \$1.10 short; 5 bills, one gold coin, and 24 silver coins make \$1 too much, and so do 5 gold coins, 20 silver, 100 copper, and one bill. What were his different kinds of money worth, and what was his railroad fare?

CHAPTER VII.

THE SECOND METHOD OF ELIMINATION.

Linear and Quadratic Pairs.

177. Model A.—A certain rectangular field, containing 480 square rods, requires 104 rods of fence to enclose it. What are its dimensions?

Let $x =$ length in rods;

then $\dfrac{480}{x} =$ breadth in rods.

①	$2x + 2\dfrac{480}{x} = 104$	
②	$x^2 + 480 = 52x$	① $\times\, x \div 2$
③	$x^2 - 52x + 480 = 0$	② $- 52x$

and so on. Or otherwise,

Let $x =$ length in rods;

then $52 - x =$ breadth in rods.

①	$x(52 - x) = 480$	
②	$52x - x^2 = 480$	same as ①
③	$x^2 - 52x + 480 = 0$	② $- 52x + x^2$

and so on.

Still another way would be to use two unknown letters, as follows:

Let x = length in rods; y = breadth in rods.

$$
\begin{array}{lll}
① & xy = 480 & \\
② & 2x + 2y = 104 & \\
③ & x + y = 52 & ② \div 2
\end{array}
$$

Equations ① and ③ we cannot solve by the method of elimination heretofore tried. We can find the value of x from the first equation,—not the numerical value that satisfies both ① and ③, but a sort of formula for x,—and then substitute in the other equation; that is, write instead of x the formula for it. Then, if the formula for x had *no letter x in it*, we shall have an equation from which x has been eliminated. Thus,—

$$
\begin{array}{lll}
④ & x = \dfrac{480}{y} & ① \div y \\
⑤ & \dfrac{480}{y} + y = 52 & ④ \text{ substituted in } ③ \\
⑥ & 480 + y^2 = 52y & ⑤ \times y \\
⑦ & y^2 - 52y + 480 = 0 & ⑥ - 52y
\end{array}
$$

and so on. Or otherwise,

$$
\begin{array}{lll}
④ & x = 52 - y & ③ - y \\
⑤ & y(52 - y) = 480 & ④ \text{ substituted in } ① \\
⑥ & 52y - y^2 = 480 & \text{same as } ⑤ \\
⑦ & 0 = y^2 - 52y + 480 & ⑥ - 52y + y^2
\end{array}
$$

Or we could in the same way have eliminated y; and the rest of the solution would have been identical with the solutions with one unknown letter which were worked out first.

The use of the second unknown letter may often, as in this case, be a way of explaining how to get with one letter abbreviations for the two unknown numbers.

Model B.—A cistern can be filled by two pipes, running together, in 2 hours 55 minutes; the larger pipe by itself will fill it 2 hours sooner than the smaller pipe by itself. How long for each pipe separately?

$x =$ number of hours for smaller pipe;
$y =$ number of hours for larger pipe.

(1) $x - y = 2$	
(2) $\frac{1}{x} + \frac{1}{y} = \frac{12}{35}$	
(3) $35y + 35x = 12xy$	(2) $\times\ 35xy$
(4) $x = y + 2$	(1) $+\ y$
(5) $35y + 35(y+2) = 12y(y+2)$	(4) subst. in (2)
(6) $12y^2 + 24y = 70y + 70$	same as (5)
(7) $12y^2 - 46y - 70 = 0$	(6) $- 70y - 70$
(8) $6y^2 - 23y - 35 = 0$	(7) $\div\ 2$
(9) $(6y + 7)(y - 5) = 0$	(8) factored

$\therefore\ y = 5$, or $-\frac{7}{6}$

5 hours for larger, 7 for smaller. *Ans.*

Model C.—A number of two digits is 9 less than the square of the sum of its digits; and if 45 be subtracted from the number, the digits are reversed.

Let $x =$ the tens digit; $y =$ the units digit.

(1) $10x + y + 9 = x^2 + 2xy + y^2$	
(2) $10x + y - 45 = 10y + x$	
(3) $9x - 9y = 45$	(2) $- 10y - x$
(4) $x - y = 5$	(3) $\div\ 9$
(5) $x = y + 5$	(4) $+\ y$
(6) $10(y+5) + y + 9$ $= y^2 + 10y + 25 + 2y(y+5) + y^2$	(5) subst. in (1)
(7) $11y + 59 = 4y^2 + 20y + 25$	same as (6)
(8) $4y^2 + 9y - 34 = 0$	(7) $- 11y - 59$
(9) $(4y + 17)(y - 2) = 0$	(8) factored

$\therefore\ y = 2;\ y = -\frac{17}{4}$

$x = y + 5 = 7$ *Ans.* 72

In the above examples, the equations of the first degree,* since they may be represented by a straight line in a diagram, are called **linear** equations. The other equation is of the second degree,* and is called a quadratic. The rule for this method of elimination (called **Elimination by Substitution**) is :

178. Find the formula for x (or for y) **from the linear equation, and substitute in the quadratic.**

EXERCISE LXXXIX.

1. $x + y = 9;\ \frac{1}{x} + \frac{1}{y} = \frac{1}{2}.$
2. $xy = 1;\quad 3x - 5y = 2.$
3. $x - \frac{x - y}{2} = 4;\ y - \frac{x + 3y}{x + 2} = 1.$
4. $7x^2 - 8xy = 159;\quad 5x + 2y = 7.$
5. $x^2 - 2xy - y^2 = 31;\ x + y = 13.$
6. $3x - 10y = 1;\ x^2 - xy = 5y^2 + 79.$
7. $x + y = 7;\ x^2 + 2y^2 = 34.$
8. $x = y - 2;\ 3x^2 = 4y^2 + 48.$
9. $3y - 2x = 6;\ 4x^2 = 9(y^2 + 12).$
10. $x^2 + 24xy = (3y + 4)^2 + 8;\ x = 3y + 2.$
11. $\frac{x}{2} + \frac{y}{3} = 1;\ \frac{2}{x} + \frac{3}{y} = 4.$
12. $3x^2 - 2xy = 15;\ 2x + 3y = 12.$
13. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2};\ 3x - 2y = 16.$
14. $11y + 6x = 62;\ x^2 - xy + y^2 = 13.$
15. $\frac{2x}{3} - \frac{11y}{6} = \frac{1}{2};\ x + y + 3xy = 93.$

* The degree of an equation is the degree of its highest term ; and the degree of a term is the number of unknown letters that are factors of it ; thus $3x^2y^3$ would be of the fifth degree, because it contains two factors x and three factors y. See § 75.

16. $\dfrac{x+y}{2}-\dfrac{x-y}{3}=\dfrac{7}{6};\ x^2+6xy+2y^2+4x+3y=29.$

17. $10-\dfrac{2x+3y}{2}=\dfrac{x+y}{3};\ 3x^2-6x+5y=20.$

18. $\dfrac{x+y}{x-y}=2\tfrac{1}{2};\ \dfrac{x}{y}-\dfrac{3x}{x+y}=\dfrac{7}{2(x+y)}.$

19. $\dfrac{x+y+1}{x-y-1}=2;\ 3x=\dfrac{x^2}{y}-27.$

20. $\dfrac{2y+x-1}{9}=\dfrac{x+y}{7};\ x^2+4xy=3y^2+20x+6.$

21. Find two numbers whose sum is 4 times their difference, and the difference of whose squares is 196.

22. A path around a rectangular garden is 7 feet wide and 1806 feet in area, which is 994 feet less than the area of the garden itself. Find the size of the garden.

23. A number is 7 times the sum of its two digits; and if the number is multiplied by its first digit, the product is 672. Find the number.

24. The difference of two numbers is $\frac{5}{12}$ of the greater, and the difference of their squares is 380. Find the numbers.

25. If a certain rectangular field were 75 rods longer and 20 rods wider, its length would be double its breadth, and its area would be double what it is now. Find its dimensions.

26. There are in a certain block of exactly similar houses 300 rooms; 5 more houses in the block than there are rooms in one of the houses. How many houses in the block?

27. The front wheel of a bicycle makes 16 turns less than the hind wheel in going a mile; if the front wheel were 6 inches more in circumference, it would turn 60 times less than the hind wheel in going a mile. Find the circumference of each wheel.

28. When a certain train has travelled 5 hours it is still 60 miles short of its terminus; and on the whole trip 1 hour can be saved by running 5 miles an hour faster. Find speed of train and length of trip.

29. A and B start on a road race together; A is a sure winner, and looking back once on the road he sees B 60 rods behind. A crosses the line 4 minutes after that, and B comes in 9 minutes behind. When A looked back he had as far to ride as B had already ridden. Find their speeds.

30. The fore wheel of a carriage turns 132 times more than the hind wheel in going a mile; and 6 turns of the fore wheel cover 2 feet less than 5 turns of the hind wheel. Circumferences?

31. Rowed 24 miles down-stream and back again; took 8 hours longer on the return trip, the current reducing the speed to $\frac{1}{3}$ of what it was going down. Rate of the current?

32. Two trains start from opposite ends of a double-track railroad 300 miles long; after they pass, one train takes 9 hours, the other 4, to complete the journey. Speed of each train?

33. If 1 is added to the denominator of a fraction, the value of the fraction becomes $\frac{1}{3}$; and if 2.1 be added to the fraction, the fraction is inverted. Find the fraction.

34. If 2 be added to the denominator of a certain fraction, its value becomes $\frac{1}{2}$; if 4 be added to each term of the fraction, its value is increased by $\frac{1}{8}$. Find the fraction.

35. A rectangular field 238 rods in area loses 58 square rods by taking off a strip 1 rod wide all around the edge for roads. Find the dimensions of the field that remains.

36. Five leaks and a drain-pipe empty a cistern in 4 hours; the average time for one of the leaks to empty it would be 24 hours more than half the time the drain-pipe

requires. How long will it take the cistern to be emptied if the leaks are stopped?

37. When a certain kind of cloth is wetted it shrinks up $\frac{1}{8}$ in length and $\frac{1}{16}$ in width. A piece of this cloth used for an awning was found to have shrunk $8\frac{5}{8}$ square yards in area, while the edging bought for it before it shrank, intended to go on one long side and both ends, was found to be 2 yards too long. What was the length and the width of the cloth before shrinking?

38. The area of a rectangular plot of land is 16 feet less than a square plot of equal perimeter; and its breadth is one foot more than $\frac{5}{6}$ of the length. Find the dimensions of the rectangular plot.

39. Invested in 5% stocks; then invested the same sum in 6% stocks, paying \$6 higher premium and getting 2 shares less; the income from the second investment is \$18 more than that from the first. What was the sum invested?

40. Two boys run in opposite directions around a rectangular field the area of which is one acre; they start at one corner and meet 13 yards from the opposite corner; and one of the boys could go around the field 6 times while the other is going 5 times. Find the dimensions of the field.

PAIRS OF QUADRATICS.

179. If we were to make diagrams for the equations in this chapter as we did for some of the equations in the preceding chapter, we should find that the points for an equation of the second degree formed a curved line,—a circle or an ellipse, or perhaps one of the two kinds of curves shown in Fig. 8,—while the points for an equation of the first degree formed a straight line.

180. A straight line can cut one of these curves twice; that is, there are two points which represent an answer to both equations, and consequently we always expect two answers when we have one equation of the first degree and one of the second. When both equations are of the second degree the case is different: then both of the lines which represent the equations are curved lines, and they can cross

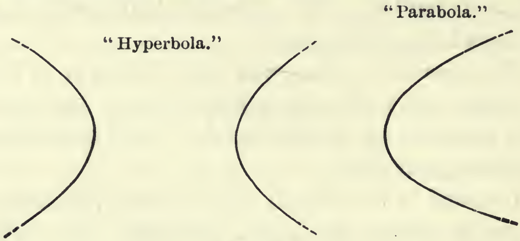


FIG. 8.

each other four times, as in Fig. 7, Chapter VI. In other words, **two equations of the second degree have four answers,**—four sets of particular values for x and y that satisfy both of the given equations.

181. This very fact,—that there are four answers, and consequently that the equation we should get after substituting would have four factors,—is enough to show us that **we cannot GENERALLY solve a pair of quadratic equations**, because we cannot GENERALLY factor an equation of the fourth degree.

182. Special cases arise, however, which can be solved by some special method,—sometimes requiring a great deal of ingenuity.

Only Two Kinds of Terms.

183. In the following set of examples, since there are only two kinds of terms that are not similar, the method of combination may be used:

EXERCISE XC.

1. $\begin{cases} 2x^2 + 3y^2 = 98 \\ x^2 + 2y^2 = 57 \end{cases}$

*2. $\begin{cases} 3x^2 - 5xy = 7 \\ 4xy - 2x^2 = 14 \end{cases}$

3. $\begin{cases} 4x^2 + 7y^2 = 148 \\ 3x^2 - y^2 = 11 \end{cases}$

*4. $\begin{cases} 3xy + 5y^2 = 155 \\ 7y^2 - 16xy = 15 \end{cases}$

*5. $\begin{cases} 15x^2 + 4xy = 40 \\ 7xy - 12x^2 = 2 \end{cases}$

FINDING A LINEAR EQUATION.

184. The first effort of a pupil, in trying to solve a pair of quadratics, should be to work out from them, if possible, some equation of the first degree, which, with either of the original quadratics, will form a LINEAR AND QUADRATIC PAIR.

Model D.

(1) $2x^2 - 5x + 3y = 171$	
(2) $3x^2 + 3x - 13y = 239$	
(3) $6x^2 - 15x + 9y = 513$	(1) $\times 3$
(4) $6x^2 + 6x - 26y = 478$	(2) $\times 2$
(5) $21x - 35y = 35$	(4) $-$ (3)
(6) $3x - 5y = -5$	(5) $\div 7$
(7) $y = \dfrac{3x + 5}{5}$	[(6) $+ 5 + 5y$] $\div 5$
(8) $2x^2 - 5x + 3\,\dfrac{3x + 5}{5} = 171$	(7) substituted in (1)
(9) $10x^2 - 25x + 9x + 15 = 855$	(8) $\times 5$
(10) $10x^2 - 16x - 840 = 0$	(9) $- 855$
(11) $5x^2 - 8x - 420 = 0$	(10) $\div 2$
(12) $(5x + 42)(x - 10) = 0$	
$\therefore x = 10$ or $-\frac{42}{5}$	

* See footnote, p. 172.

In this example we can get only one equation of the first degree to substitute from; and consequently here (that is, where the terms of the second degree are alike in any pair of quadratics*) we can get only two sets of answers.

The values of y are obtained by substituting the values of x successively in the equation of the first degree by which they were obtained (in this case ⑥).

$$\begin{array}{rlrl} 30 - 5y &= -5 & -\frac{126}{5} - 5y &= -5 \\ 5y &= 35 & 126 + 25y &= 25 \\ y &= 7 & 25y &= -101 \\ & & y &= -\frac{101}{25} \end{array}$$

185. In order to show what values of x and y belong to the same answer, it is best to arrange the results in a table, thus:

x	10	$-\frac{42}{5}$
y	7	$-\frac{101}{25}$

186. We might also have solved by eliminating y:

③	$26x^2 - 65x + 39y = 2223$	① $\times$ 13
④	$9x^2 + 9x - 39y = 717$	② $\times$ 3
⑤	$35x^2 - 56x = 2940$	③ + ④
⑥	$5x^2 - 8x - 420 = 0$	⑤ $\div$ 7 $-$ 420

whence as before.

This method can be used when in both equations all the terms containing one of the unknown letters are similar.

* In fact, whenever the expressions made up of terms of the second degree in each equation have a common factor containing x or y.

One Quadratic Factorable.

187. Sometimes one of the given equations can be factored:

Model E.

(1) $y^2 - 4xy + 8y = 16x - 7 - 4x^2$	
(2) $x^2 + 2xy = 24$	
(3) $4x^2 - 4xy + y^2 - 16x + 8y + 7 = 0$	(1) $+ 7 + 4x^2 - 16x$
(4) $(2x - y)^2 - 8(2x - y) + 7 = 0$	same as (3)
(5) $(2x - y - 7)(2x - y - 1) = 0$	(4) factored
(6) $2x - y - 7 = 0$	(5) Ax. A
(7) $y = 2x - 7$	(6) $+ y$
(8) $2x - y - 1 = 0$	(5) Ax. A
(9) $y = 2x - 1$	(8) $+ y$
(10) $x^2 + 2x(2x - 7) = 24$	(7) subst. in (2)
(11) $5x^2 - 14x - 24 = 0$	(8) $- 24$
(12) $(5x + 6)(x - 4) = 0$	(11) factored
$\therefore x = -\frac{6}{5};\quad x = 4$	
(13) $x^2 + 2x(2x - 1) = 24$	(9) subst. in (2)
(14) $5x^2 - 2x - 24 = 0$	(13) $- 24$
(15) $(5x - 12)(x + 2) = 0$	(14) factored
$x = \frac{12}{5};\ x = -2$	

Substituting these four values of x, **EACH IN THE LINEAR EQUATION BY WHICH IT WAS OBTAINED**, we get the following four answers:

x	$-\frac{6}{5}$	4	$\frac{12}{5}$	-2
y	$-\frac{47}{5}$	1	$\frac{19}{5}$	-5

Of these four answers the first two depend on (7), and the others on (9).

One Quadratic Homogeneous.

188. The method just described can be applied whenever one equation of the pair of quadratics is **homogeneous**, that is, where every term is of the same degree. A homogeneous equation with two unknown letters is also called a **binary** equation.

Model F.

(1) $x^2 + xy + 2y^2 = 74$	
(2) $24y^2 - 25xy = 25x^2$	
(3) $25x^2 + 25xy - 24y^2 = 0$	(2) $+ 25xy - 24y^2$
(4) $(5x + 8y)(5x - 3y) = 0$	(3) factored
(5) $5x + 8y = 0$	(4) Ax. A
(6) $y = -\frac{5x}{8}$	[(5) $- 5x] \div 8$
(7) $5x - 3y = 0$	(4) Ax. A
(8) $y = \frac{5x}{3}$	[(7) $+ 3y] \div 3$
(9) $x^2 - \frac{5x^2}{8} + \frac{50x^2}{64} = 74$	(6) substituted in (1)
(10) $74x^2 = 74 \times 64$	(9) $\times 64$
(11) $x^2 - 64 = 0$	(10) $\div 74 - 64$
$\therefore x = 8;\ x = -8$	(11) Ax. A
(12) $x^2 + \frac{5x^2}{3} + \frac{50}{9} = 74$	(8) substituted in (1)
(13) $74x^2 = 74 \times 9$	(12) $\times 9$
(14) $x^2 - 9 = 0$	(13) $\div 74 - 9$
$x = 3;\ x = -3$	

Remembering that the first two values of x were found on the supposition that $y = -\frac{5x}{8}$, and the others on (8), we get for our table of answers

x	8	-8	3	-3
y	-5	5	5	-5

189. The general principle of finding a linear equation from the pair of quadratics, either by eliminating one kind of terms or by factoring one of the equations, is illustrated by the following examples.

EXERCISE XCI.

Find four answers wherever possible:

1. $\begin{cases} 3x^2 - 2x + 5xy = 96 \\ 2xy - 5x = 15 \end{cases}$ **2.** $\begin{cases} x^2 - 40 = 2y^2 + 21y \\ x^2 - 10x + 25 = 4y^2 \end{cases}$

3. $\begin{cases} 4x^2 - 8xy + 3y^2 - 8y + 29 = 0 \\ 2x^2 + y^2 = 4xy - 7 \end{cases}$

4. $\begin{cases} 3x^2 - 7xy + y^2 + 9y + 66 = 0 \\ 3x^2 + 28y^2 + 48 = 7xy \end{cases}$

5. $\begin{cases} x^2 + xy + y^2 = 19 \\ 6x^2 - 13xy + 6y^2 = 0 \end{cases}$ **6.** $\begin{cases} x^2 + 3xy = 54 \\ 115x^2 + 291xy = 216y^2 \end{cases}$

7. $\begin{cases} 7(x + y^2) - 22y = 5xy + x^2 + 16 \\ x^2 + 5xy - 6y^2 = 0 \end{cases}$

8. $\begin{cases} xy + 24 = 0 \\ 6x^2 + 13xy + 6y^2 = 0 \end{cases}$ **9.** $\begin{cases} 2x^2 + 27y^2 = 21xy \\ x^2 + 3xy = 27 \end{cases}$

10. $\begin{cases} xy = y^2 - 4 \\ x^2 + 9x = -3xy \end{cases}$

11. $18x^2 - 9xy = 14x^2 + 28y^2$; $x^2 + 2y^2 = 18$.

12. $16x^2 + 8xy = 15x^2 - 15y^2$; $2x^2 + xy = 15$.

13. $24x^2 + 13xy = 2y^2$; $x^2 + xy - y^2 = -11$.

14. $4x^2 + 12y^2 = 7x^2 + 7xy + 14y^2$; $x^2 + 3y^2 = 28$.

15. $69x^2 + 23xy - 69y^2 = 66x^2 - 33y^2$; $2x^2 - y^2 = 23$.

16. $4x^2 + 20xy - 4y^2 = 7x^2 + 21xy - 14y^2$; $2y^2 - x^2 - 3xy = 4$.

17. $11x^2 - 11xy - 176y^2 = 2x^2 + 2xy - 16y^2$; $x^2 + xy = 22 + 8y^2$.

18. $5x^2 + 5y^2 = 29x^2 - 58xy$; $x^2 - 2xy = 5$.

19. $-3x^2 + 6xy = 32xy + 16y^2$; $x^2 - 2xy = 16$.
20. $15x^2 + 45xy = -7xy - 28y^2$; $xy + 4y^2 = 30$.

190. The two equations obtained by factoring a binary equation are either inconsistent or identical; and further, two binary equations can never be simultaneous, for the same reason. *Find out why.*

Finding a Binary Equation.

191. When in each equation the terms containing the unknown letters are all of the second degree, it is possible to get a homogeneous equation by **eliminating the numerical terms:**

Model G.

1	$x^2 + y^2 = 5$	
2	$2x^2 + xy + y^2 = 8$	
3	$8x^2 + 8y^2 = 40$	1 $\times 8$
4	$10x^2 + 5xy + 5y^2 = 40$	2 $\times 5$
5	$2x^2 + 5xy - 3y^2 = 0$	4 $-$ 3
6	$(2x - y)(x + 3y) = 0$	5 factored
7	$2x - y = 0$	6 Ax. A
8	$y = 2x$	7 $+ y$
9	$x + 3y = 0$	6 Ax. A
10	$y = -\frac{x}{3}$	[9 $- x$] $\div 3$
11	$x^2 + 4x^2 = 5$	8 substituted in 1
12	$x^2 - 1 = 0$	11 $\div 5 - 1$
13	$(x + 1)(x - 1) = 0$	12 factored
14	$x + 1 = 0 \therefore x = -1$	13 Ax. A
15	$x - 1 = 0 \therefore x = 1$	13 Ax. A
16	$x^2 + \frac{x^2}{9} = 5$	10 substituted in 1
17	$10x^2 = 45$	16 $\times 9$
18	$x^2 - \frac{9}{2} = 0$	17 $\div 10 - \frac{9}{2}$

Since the square root of $\frac{9}{2} = 2.121+$, we can continue the work as follows:

⑲ $(x + 2.121)(x - 2.121) = 0$ ⑱ factored
⑳ $x + 2.121 = 0$; $x = -2.121$ ⑲ Ax. A
㉑ $x - 2.121 = 0$; $x = 2.121$ ⑲ Ax. A

192. The values of x given in ⑭ and ⑮ were obtained on the supposition that ⑧ was true, that is, that $y = 2x$; and the other values of x were obtained on the supposition that ⑩ was true, that is, that $y = -\frac{x}{3}$; consequently, we must be careful to substitute ⑭ and ⑮ in ⑧ [NOT in one of the original equations], and ⑳ and ㉑ in ⑩. In this way we get

x	1	-1	2.121	-2.121
y	2	-2	$-.707$	.707

Instead of figuring out the square root of $\frac{9}{2}$ we might have represented it by the symbol $\sqrt{\frac{9}{2}}$; our table of answers would then have been

x	1	-1	$\sqrt{\frac{9}{2}}$	$\sqrt{\frac{9}{2}}$
y	2	-2	$-\frac{1}{3}\sqrt{\frac{9}{2}}$	$\frac{1}{3}\sqrt{\frac{9}{2}}$

193. Whenever, as in this case, the root of a number cannot be exactly expressed, the symbol that stands for its exact value is called a **surd,** or **irrational** quantity. Later we shall learn methods of reducing and combining such quantities.

EXERCISE XCII.

1. $\begin{cases} x^2 + xy + y^2 = 28 \\ 2xy + y^2 - x^2 = 28 \end{cases}$

2. $\begin{cases} 3y^2 - 5x^2 = 63 \\ x^2 + xy = 27 \end{cases}$

3. $\begin{cases} x^2 + 3xy + 2y^2 = 40 \\ 2x^2 - 2xy + y^2 = 5 \end{cases}$

4. $\begin{cases} x^2 + 9xy = 340 \\ 7xy - y^2 = 171 \end{cases}$

5. $\begin{cases} x^2 + xy + 2y^2 = 74 \\ 2x^2 + 2xy + y^2 = 73 \end{cases}$

6. $\begin{cases} 5x^2 + 3xy + 2y^2 = 188 \\ x^2 - xy + y^2 = 19 \end{cases}$

7. $\begin{cases} 8x^2 - 3xy - y^2 = 40 \\ 9x^2 + xy + 2y^2 = 60 \end{cases}$

8. $\begin{cases} x^2 - xy + y^2 = 7 \\ 3x^2 + 13xy + 8y^2 = 162 \end{cases}$

9. $\begin{cases} 2x^2 + 3xy = 26 \\ 3y^2 + 2xy = 39 \end{cases}$

10. $\begin{cases} x^2 - 2xy = 21 \\ xy + y^2 = 18 \end{cases}$

11. $\begin{cases} 2x^2 + xy + x = 27 \\ 9y^2 + 2x^2 = 9xy \end{cases}$

12. $\begin{cases} 2x^2 - 5xy + 2y^2 = 27 \\ x^2 + 3y^2 = 28 \end{cases}$

13. $\begin{cases} x^2 + 7y^2 = 32 \\ 3xy + 6y^2 - 4 = 32 \end{cases}$

14. $\begin{cases} 15y^2 - 2xy = 111 \\ 12(xy - y^2) = 3x^2 - xy \end{cases}$

15. $\begin{cases} 3xy + 6y^2 = 4y + 40 \\ x^2 + 6y^2 = 5xy \end{cases}$

16. $\begin{cases} x^2 + 3xy = 40 \\ x^2 + 3y^2 = 28 \end{cases}$

17. $\begin{cases} 6x^2 + 31xy - 15y^2 = 66 \\ 5x^2 - y^2 = 5 \end{cases}$

18. $\begin{cases} 2x^2 - xy - 6y^2 = 23 \\ xy = 21 \end{cases}$

19. $\begin{cases} 2x^2 + 11xy = 120 \\ x^2 - 30y^2 = 105 \end{cases}$

20. $\begin{cases} x^2 + xy = 70 - 2x \\ \dfrac{x}{x - y} = \dfrac{10(x + y)}{3(x - 3y)} \end{cases}$

The "Symmetrical" Method.

194. Where both equations are symmetrical,—that is, where x and y enter into each in such a way that if they were interchanged the resulting equation would be identical with the given one,—a method of combination is employed to reduce the two equations to the form $x+y=\ldots$ and $x-y=\ldots$, as illustrated by the following examples:

Model H.

$$
\begin{array}{lll}
① & x+y=15 & \\
② & xy=36 & \\
\hline
③ & x^2+2xy+y^2=225 & ①^2 \\
④ & 4xy=144 & ② \times 4 \\
⑤ & x^2-2xy+y^2=81 & ③ - ④ \\
⑥ & (x-y)^2-81=0 & ⑤ - 81 \\
⑦ & x-y=9; \quad ⑧\ x-y=-9 & ⑥\ \text{Ax. A} \\
⑨ & 2x=24 & ① + ⑦ \\
⑩ & 2y=6 & ① - ⑦ \\
⑪ & 2x=6 & ① + ⑧ \\
⑫ & 2y=24 & ① - ⑧
\end{array}
$$

x	12	3	*Ans.*
y	3	12	

Model I.

$$
\begin{array}{lll}
① & x^2+y^2=74 & \\
② & xy=35 & \\
\hline
③ & x^2+2xy+y^2=144 & ① + 2 \times ② \\
④ & x^2-2xy+y^2=4 & ① - 2 \times ② \\
⑤ & (x+y)^2-144=0 & ③ - 144
\end{array}
$$

⑥ $(x-y)^2-4=0$ ④ -4
⑦ $x+y=12$; ⑧ $x+y=-12$ ⑤ Ax. A
⑨ $x-y=2$; ⑩ $x-y=-2$ ⑥ Ax. A
⑪ $2x=14$ ⑦ + ⑨
⑫ $2y=10$ ⑦ − ⑨
⑬ $2x=-10$ ⑧ + ⑨
⑭ $2y=-14$ ⑧ − ⑨
⑮ $2x=10$ ⑦ + ⑩
⑯ $2y=14$ ⑦ − ⑩
⑰ $2x=-14$ ⑧ + ⑩
⑱ $2y=-10$ ⑧ − ⑩

x	7	-5	5	-7
y	5	-7	7	-5

Ans.

Model J.

① $x^2+y^2=185$
② $x+y=17$

③ $x^2+2xy+y^2=289$ ②^2
④ $2xy=104$ ③ − ①
⑤ $x^2-2xy+y^2=81$ ① − ④
⑥ $(x-y)^2-81=0$ ⑤ -81
⑦ $x-y=9$; ⑧ $x-y=-9$ ⑥ Ax. A
⑨ $2x=26$ ② + ⑦
⑩ $2y=8$ ② − ⑦
⑪ $2x=8$ ② + ⑧
⑫ $2y=26$ ② − ⑧

x	13	4
y	4	13

Ans.

195. Sometimes the equations are symmetrical except that the interchange of variables produces a change of sign. The general method illustrated above is still applicable. The same interchange can of course be made in the set of simple equations that constitute the answers; and, consequently, in these as in all examples that can be solved by the symmetrical method, it is only necessary to find half the answers,—the others can be written down by making the interchange of x and y.

Model K.

(1) $x - y = 3$
(2) $x^2 + y^2 = 65$

(3) $x^2 - 2xy + y^2 = 9$	(1)^2
(4) $2xy = 56$	(2) − (3)
(5) $x^2 + 2xy + y^2 = 121$	(2) + (4)
(6) $(x + y)^2 - 121 = 0$	(5) − 121
(7) $x + y = 11$; (8) $x + y = -11$	(6) Ax. A
(9) $2x = 14$	(7) + (1)
(10) $2y = 8$	(7) − (1)
(11) $2x = -8$	(8) + (1)
(12) $2y = -14$	(8) − (1)

x	7	− 4
y	4	− 7

Ans.

A Short Cut.

196. When a quadratic equation is prepared for factoring, if the first straight product is x^2, the second straight product is THE PRODUCT OF THE TWO ROOTS; and the sum of the cross products will be $-x$ multiplied by THE SUM OF THE ROOTS.* For instance, the equation $x^2 - 7x + 12 = 0$ has answers $x = 3$ and $x = 4$.

* See footnote, p. 124.

Now this property of the quadratic equation can be made use of in solving symmetrical equations; the values of xy and $x + y$, or of xy and $x - y$, are early obtained; and remembering in what cases the two values of x are opposite in sign, we shall know when to make the second straight product negative:

Model L.

1 $x + y = 13$
2 $xy = 36$

3 $x^2 - 13x + 36 = 0$ *Auxiliary quadratic.*
4 $x = 9$; $x = 4$ 3 Ax. A

x	9	4
y	4	9

Model M.

1 $x - y = 7$ or $x - y = -7$
2 $xy = 18$
3 $x^2 - 7x - 18 = 0$ or $x^2 + 7x - 18 = 0$ *Aux. quad.*
4 $\left.\begin{array}{l} x = 9 \\ x = -2 \end{array}\right\}$ or $\left.\begin{array}{l} x = -9 \\ x = 2 \end{array}\right\}$ 3 Ax. A

x	9	-2	-9	2
y	2	-9	-2	9

Ans.

197. The auxiliary quadratic constructed in this way is precisely the same as would be obtained by the method of substitution.

Irregular Devices.

198. Equations of higher degrees than quadratics can sometimes be solved. Often considerable ingenuity is required to get a solution; and it must be remembered

that it is only an accident when a pair, even of quadratics, is solvable at all by elementary algebra.

One device of frequent use is to divide one equation by the other.

Another device that is often of great service may be applied to example 14, below. Let z stand for $x^2 + y^2$ and u for $2xy$. The two equations then become $3z - u = 27$; $4z - 3u = 16$. When u and z are found, equate their values to $2xy$ and $x^2 + y^2$ respectively and the rest of the way is familiar.

EXERCISE XCIII.

In the following examples occasional hints are given to enable the student to choose a good method of elimination:

1. $x - y = 8$; $xy = 33$.
2. $x^2 + y^2 = 68$; $xy = 16$.
3. $x^3 - y^3 = 37$; $x^2 + xy + y^2 = 37$.
4. $x^3 + y^3 = 152$; $x^2 - xy + y^2 = 19$.
5. $x^3 + y^3 = 407$; $x + y = 11$.
6. $x^3 - y^3 = 2197$; $x - y = 13$.
7. $x^4 + x^2y^2 + y^4 = 2128$; $x^2 + xy + y^2 = 76$.
8. $\dfrac{x}{y} + \dfrac{y}{x} = 2\frac{1}{2}$; $x + y = 6$.
9. $\dfrac{34}{x^2 + y^2} = \dfrac{15}{xy}$; $x + y = 8$.
10. $x^2y^2 + xy = 42$; $x + y = 5$.
11. $2x + y = x^2$; $2y + x = y^2$. [Subtract and factor.]
12. $x^2 + xy = 210$; $y^2 + xy = 231$. [Add.]
13. $xy(x + y) = 30$; $x^3 + y^3 = 35$. [Divide and cancel.]
14. $3(x^2 + y^2) - 2xy = 27$; $4(x^2 + y^2) - 6xy = 16$. [Solve for $x^2 + y^2$ and $2xy$.]
15. $x^3 + 4xy + y^3 = 38$; $x + y = 2$. [Divide.]
16. $\dfrac{x + y}{7} = \dfrac{8}{x + y + 1}$; $xy = 12$. [Solve first for $x + y$.]

17. $x^3 + y^3 = \frac{27xy}{2}$; $x + y = 9$. [Divide.]

18. $x^3 + y^3 = \frac{35x^2y^2}{36}$; $x + y = \frac{5xy}{6}$

[① ÷ ② − ②^2; solve for xy and substitute.]

19. $x^2 + y^2 = xy + 13$; $x + y = xy - 5$.

[① + $2xy$; solve for xy and $x + y$.]

20. $x^2 + y^2 = 74$; $10(x + y) = 7xy - 5$.

[① + $\frac{2}{7}$②; solve for $x + y$.]

Symmetry not Obvious.

199. The following examples may be solved by the symmetrical method; for though they are NOT symmetrical in x and y, they are symmetrical in functions* of x and y; just as the equations of Model M are symmetrical in x and $-y$. For convenience we may adopt the same device as in example 14 above, letting z and w stand for the functions of x and y.

Model N. $9x^2 - 3xy^2 + y^4 = \frac{27x^3 + y^6}{10} = 28.$

Let $z = 3x$; $w = y^2$; then the equations become

$$z^2 - zw + w^2 = 28$$
$$z^3 + w^3 = 280$$

whence we obtain $z = 6$ or 4; $w = 4$ or 6.

If $3x = 6$, $x = 2$; If $y^2 = 4$, $y = \pm 2$;
if $3x = 4$, $x = \frac{4}{3}$. if $y^2 = 6$, $y = \pm \sqrt{6}$.

x	2	2	$\frac{4}{3}$	$\frac{4}{3}$
y	2	-2	$\sqrt{6}$	$-\sqrt{6}$

* A FUNCTION of x is an expression containing x; e.g., $2x - 1$, $3x^2 + 7$, etc.

EXERCISE XCIV.

1. $\begin{cases} 4x^2 + 9y^2 = 181 \\ xy = 15 \end{cases}$

2. $\begin{cases} x^2 + 2xy^2 + 4y^4 = 273 \\ x^3 - 8y^6 = 819 \end{cases}$

3. $\begin{cases} x^3 + 3y^2 = 39 \\ x^6 + 9y^4 = 873 \end{cases}$

4. $\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} = 7 \\ xy = 1 \\ [\text{Let } z = 3x;\ w = 4y.] \end{cases}$

5. $\begin{cases} \dfrac{16}{x^2} + 9y^4 = 13 \\ \dfrac{2y^2}{x} = 1 \\ \left[\text{Let } z = \dfrac{4}{x};\ w = 3y^2.\right] \end{cases}$

6. $\begin{cases} x + y^2 = 60 \\ x^2 + y^4 = 2522 \end{cases}$

7. $\begin{cases} \dfrac{1}{3x} + \dfrac{2}{5y} = 2 \\ 6x + 5y = 4 \end{cases}$

8. $\begin{cases} x^2 + 4y^2 = 4y + 99 \\ 2xy - x = 48 \\ [\text{Let } z = 2y - 1.] \end{cases}$

9. $\begin{cases} x^2 + 9y^2 - 12y = 9 \\ 3xy - 2x = 6 \\ [\text{Let } z = 3y - 2.] \end{cases}$

10. $\begin{cases} 4x^2 + 9y^2 + 12x - 6y = 175 \\ 2x - 9y - 6xy = 85 \\ [\text{Let } z = 2x + 3; \\ w = 3y - 1] \end{cases}$

Elimination by Comparison.

200. Still a third method of elimination, called "comparison," is recognized by students of algebra, and is sometimes of great convenience, though it is never necessary. It consists in finding an expression (a "formula") for y (or for x) from each equation and setting them equal.

Model O.

① $7y + 29 = 10x$

② $x^2 + 26 = 3xy + 2y$

③ $y = \dfrac{10x - 29}{7}$ $\qquad$ [① $- 29$] $\div 7$

④ $y = \dfrac{x^2 + 26}{3x + 2}$ $\qquad$ ② $\div (3x + 2)$

⑤ $\frac{10x - 29}{7} = \frac{x^2 + 26}{3x + 2}$	③ and ④ compared
⑥ $23x^2 - 67x - 240 = 0$	⑤ $\times\ 7(3x + 2)\ldots$
⑦ $x = 5;\ x = -\frac{48}{23}$	⑥ Ax. A
$y = 3;\ y = -\frac{509}{161}$	

This method is evidently only a slight variation of the method of substitution. Sometimes it is useful to "compare" expressions more complex than the simple x or y.

Model P.

① $x^2 + 6xy = 144$
② $6xy + 36y^2 = 432$

③ $xy + 6y^2 = 72$	② $\div\ 6$
④ $x + 6y = \frac{144}{x}$	① $\div\ x$
⑤ $x + 6y = \frac{72}{y}$	③ $\div\ y$
⑥ $\frac{144}{x} = \frac{72}{y}$	④ and ⑤ compared
⑦ $2y = x$	⑥ $\times\ xy \div 72$
⑧ $4y^2 + 12y^2 = 144$	⑦ subst. in ①
$\therefore y = \pm 3;\ x = \pm 6$	

201. This example could also have been done, like number 12 above, by adding the two equations:

③ $x^2 + 12xy + 36y^2 = 576$	① + ②
④ $x + 6y = \pm 24$	③ Ax. A.
⑤ $\pm 24x = 144$	④ subst. in ①
⑥ $\pm 144y = 432$	④ subst. in ②
$\therefore x = \pm 6;\ y = \pm 3$	

EXERCISE XCV.

Solve by comparison:

1. $y - x^2 = 2;\ y + 1 = 4x.$
2. $6y = 13x;\ y - 1 = x^2.$
3. $x^2 - 3 = \frac{y}{6};\ y + 3 = x.$
4. $y = \frac{6}{x - 1};\ y + 2 = x.$
5. $8x + \frac{7}{x} = y - 11;\ 7y = 21 + 65x.$
6. $23 - x = 7y;\ y = \frac{21}{x + 5}.$
7. $x^2 + 4 = 2xy;\ x^2 + 9 = 3xy.$
8. $x^2 + 36 = 6xy;\ 5(x - 1) = 4y.$
9. $x^2 - 5x = (x + 3)y;\ x - 3 = y - \frac{1}{x}.$
10. $y(x - 1) = x;\ 2xy = 5x - 2.$

Quadratics with Three Letters.

202. A system of three equations with three unknown letters can always be solved if two of them are of the first degree and the other is quadratic. Two of the letters must severally have their values expressed in terms of the third, and these expressions must be substituted in the quadratic equation. To this end it is in general necessary first to eliminate one of the three letters from the two equations of the first degree, and thus find one expression for substitution; and then to eliminate another letter from the same two equations, and get the other expression. But it sometimes happens, as in example 1 below, that one or both of the expressions for substitution can be found directly from the given equations.

Model Q.

①	$2x + 3y - z = 15$	
②	$x + 2y + 3z = 12$	
③	$2x^2 - yz = 48$	
④	$6x + 9y - 3z = 45$	① $\times 3$
⑤	$7x + 11y = 57$	② + ④
⑥	$11y = 57 - 7x$	⑤ $- 7x$
⑦	$y = \dfrac{57 - 7x}{11}$	⑥ $\div 11$
⑧	$4x + 6y - 2z = 30$	① $\times 2$
⑨	$3x + 6y + 9z = 36$	② $\times 3$
⑩	$11z - x = 6$	⑨ − ⑧
⑪	$z = \dfrac{6 + x}{11}$	⑩ $\div 11$
⑫	$2x^2 - \dfrac{57 - 7x}{11} \cdot \dfrac{6 + x}{11} = 48$	subst. ⑦ and ⑪ in ③
⑬	$242x^2 + 7x^2 - 15x - 342 = 5808$	⑫ $\times 121$
⑭	$249x^2 - 15x - 6150 = 0$	⑬ $- 5808$
⑮	$83x^2 - 5x - 2050 = 0$	⑭ $\div 3$
⑯	$(83x + 410)(x - 5) = 0$	⑮ factored
⑰	$x = 5;\ x = -\frac{410}{83}$	from ⑯ by Ax. A

x	5	$-\frac{410}{83}$	
y	2	$\frac{691}{83}$	*Ans.*
z	1	$\frac{8}{83}$	

EXERCISE XCVI.

1. $3x + 2y = 12;\ 3y + 2z = 11;\ x^2 + 2y^2 + 3z^2 = 25.$

2. $5x - 7y = 31;\ 3x + 7z = 27;\ 5xy - 7yz + 3xz = 51.$

3. $x^2 + 3xy - 3xz = 39;\ 2x - 3y = -9;\ 3x + 4z = 13.$

4. $3x + 2y + z = 23;\ x - 3y + 5z = 6;\ 7x^2 - 5xy = 100.$

5. $x - y + z = 4;\ 3x - 5y + 2z = 20;\ 3x^2 - 5yz = 77.$

6. $x + y + z = 9;\ 2x + 3y = 13;\ \dfrac{8}{x} + \dfrac{3}{y} = 5.$

7. $x^2 + y^2 + xz = 24;\ x + 3y + 5z = 22;\ 2y - 3z = 0.$

8. $x + y + z = 10;\ x^2 + y^2 + z^2 = 38;\ 3x + 2y - 5z = 4.$

9. $x + 3y = 11;\ 2x - z = 0;\ 3y^2 - yz = 15.$

10. $x + y + 5 = y + z + 2 = \dfrac{xy + yz}{3} = 15.$

CHAPTER VIII.

MULTIPLICATION OF FRACTIONS; HIGHEST COMMON FACTOR.

Fractions and Ratios.

203. Whenever the operation of division in Algebra cannot be carried out, or whenever for any other reason it is desirable to indicate division, the dividend may be written as a numerator, and the divisor as a denominator, and the whole result is called a **fraction**,—or, sometimes, a **ratio.**

204. It must be remembered that there are two kinds of division. For example:

The expression $\frac{30}{5}$ may mean,—

"If I divide 30 separate objects into 5 equal parts, how many separate objects will be in each part?"

Or, otherwise, the expression $\frac{30}{5}$ may mean,—

"How many equal portions, consisting each of five separate objects, may be gotten out of a collection of 30 separate objects?"

205. In the first case the divisor is an abstract number and the quotient is of the same denomination as the dividend. The indication of such division is in Algebra properly called a **fraction.**

206. In the second case the divisor is of the same denomination as the dividend, and the quotient is an abstract

number. The indication of such division is in Algebra properly called a **ratio.**

207. In multiplication the multiplier is always an abstract number, and the product is always of the same denomination as the multiplicand. In reversing the process of multiplying (that is to say, in dividing), when we get the multiplicand for our quotient we have a **fraction** of the product, and when we get the multiplier, we have the **ratio** of the product to the multiplicand.

208. Again, whenever we ask how long, or how large, or how heavy anything is, the numerical answer to our question is always a ratio,—the ratio of the quantity we are asking about to the yard or the mile, to the acre or the square inch, to the ounce, the pound, or the ton.

209. The student has probably seen by this time that Algebra does not concern itself with denominations, so that we cannot tell in any example of division whether that particular division results in a fraction or in a ratio—unless we happen to know in advance just what our letters represent. But it is not necessary to know. **All the laws of transformation are the same for fractions as for ratios.** Where it is very necessary to distinguish a ratio it may be written $a : b$ instead of $\frac{a}{b}$; but generally in Algebra such a distinction is not worth while.

210. The expression $a : b$ $\left(\text{or } \frac{a}{b}\right)$ is read "the ratio of a to b"; a is the antecedent, the dividend, or the numerator, and b is the consequent, the divisor, or the denominator, of the ratio. The form $\frac{a}{b}$ is generally read "a over b" or, more fully, "a divided by b." THE TERM FRACTION, IN THIS CHAPTER AND THE SUCCEEDING ONE, WILL BE UNDERSTOOD TO INCLUDE RATIOS.

211. The ratio of any number to ONE is the number itself; consequently any number can be regarded as having a denominator ONE.

EXERCISE XCVII.

1. The ratio of two numbers is $\frac{2}{3}$, and the smaller is 3; what is the larger?

2. The ratio of two numbers is 9, and the smaller is 3; what is the larger?

3. The sum of two numbers is 20, and their ratio is 3; what are they?

4. The sum of two numbers is 100, and their ratio is the same as the ratio of one of them to 100. Find the value of the numbers within one-tenth.

5. One number is 12 greater than the square of another, and their ratio is 7. Find the numbers.

THE LAWS OF FRACTIONS.

212. Theorems which are of use in this chapter and the succeeding one are proved by the use of the three fundamental laws of Algebra.*

213. Just as the sign $-$ means the reverse of addition, addition undone, so the sign $\div$ means the reverse of multiplication, multiplication undone. The sign $-$ may be read "the negative of," and the sign $\div$ may be read "the reciprocal of."

214. The law of association for $+$ and $-$ signs implies such identities as these:

$$a+b-c+d-e \equiv (a+b)-(c-d+e) \equiv a+(b-c)+(d-e)$$

* See § 86.

In the same way the law of association for $\div$ and $\times$ implies:

$$a \times b \div c \times d \div e \equiv (a \times b) \div (c \div d \times e) \equiv a \times (b \div c) \times (d \div e)$$

In fact, THE ASSOCIATIVE AND DISTRIBUTIVE LAWS ARE, SO FAR AS FORM GOES, THE SAME FOR $\times$ AND $\div$ AS THEY ARE FOR $+$ AND $-$.

215. Theorem I. Any fraction may have its numerator and denominator multiplied or divided by the same number without altering its value.

Let $\frac{a}{b}$ be any fraction and let m be any number. Then

$$\begin{aligned} \frac{am}{bm} &\equiv am \div bm \equiv a \times m \div (b \times m) \\ &\equiv a \times m \div b \div m \text{ [the distributive law]} \\ &\equiv a \div b \times m \div m \text{ [the commutative law]} \\ &\equiv a \div b \equiv \frac{a}{b} \end{aligned}$$

$$\therefore \frac{am}{bm} \equiv \frac{a}{b}$$

216. Theorem II. Fractions are multiplied by multiplying their numerators together for a new numerator and their denominators for a new denominator.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions.

$$\begin{aligned} \frac{a}{b} \times \frac{c}{d} &\equiv a \div b \times c \div d \equiv a \times c \div b \div d \\ &\equiv a \times c \div (b \times d) \equiv \frac{ac}{bd} \end{aligned}$$

$$\therefore \frac{a}{b} \times \frac{c}{d} \equiv \frac{ac}{bd}$$

217. Theorem III. The rule for dividing by a fraction is: **Invert the divisor and then multiply.**

Let x be any number and $\frac{a}{b}$ any fraction.

$$x \div \frac{a}{b} \equiv x \div (a \div b) \equiv x \div a \times b \equiv x \times b \div a$$

$$\equiv x \times \frac{b}{a}$$

$$\therefore\ x \div \frac{a}{b} \equiv x \times \frac{b}{a}$$

Another way of stating the rule is this: **To divide by any number, multiply by its reciprocal.**

Reduction of Multiplications.

218. The first step in every example like the following is to factor numerator and denominator in every fraction:

Model A.

$$\frac{x^2-5x-66}{x^2-5x+6} \times \frac{x^2-2x}{x^2+6x-72} \div \left(\frac{x^2-121}{x^2-9} \div \frac{x^2+5x-66}{x^2+9x+18}\right)$$

$$\equiv \frac{(x-11)(x+6)}{(x-2)(x-3)} \times \frac{x(x-2)}{(x+12)(x-6)} \div \frac{(x+11)(x-11)}{(x+3)(x-3)} \times \frac{(x+11)(x-6)}{(x+3)(x+6)}$$

$$\equiv \frac{(x-11)(x+6)}{(x-2)(x-3)} \times \frac{x(x-2)}{(x+12)(x-6)} \times \frac{(x+3)(x-3)}{(x+11)(x-11)} \times \frac{(x+11)(x-6)}{(x+3)(x+6)}$$

According to Theorem II all the factors in the numerators, as the several fractions now stand, are factors of the numerator of the product which is the answer; and the same for the denominators.

According to Theorem I we may divide the numerator and denominator of the product by the same number without altering its value; that is, we may strike out

any factor that is the same in both. The expression in Model A then becomes

$$\frac{(x-11)(x+6)}{(x-2)(x-3)} \cdot \frac{x(x-2)}{(x+12)(x-6)} \cdot \frac{(x+3)(x-3)}{(x+11)(x-11)} \cdot \frac{(x+11)(x-6)}{(x+3)(x+6)} = \frac{x}{x+12}$$

Ans.

EXERCISE XCVIII.

Perform the operations indicated:

1. $\frac{3x}{4y} \times \frac{7y}{12x} \div \frac{4x^2}{9y^2}$. 2. $\frac{a-b}{a^2+ab} \times \frac{a^2-b^2}{a^2-ab}$.

3. $\frac{x^2+x-2}{x^2-7x} \div \frac{x^2+2x}{x^2-13x+42}$.

4. $\frac{x^2+3x+2}{x^2-5x+6} \times \frac{x^2-7x+12}{x^2+x}$.

5. $\frac{x^2-4x+3}{x^2-5x+4} \cdot \frac{x^2-9x+20}{x^2-10x+21} \cdot \frac{x^2-7x}{x^2-5x}$.

6. $\frac{1}{a^2-17a+30} \div \frac{1}{a-15}$.

7. $\frac{a^2-2ay+y^2-b^2}{a^2+2ay+y^2-b^2} \div \frac{a-y+b}{a+y-b}$.

8. $\frac{a^4+a^3b-8a^2b^2+19ab^3-15b^4}{a^2+3ab-5b^2}$.

9. $\dfrac{x^4-\dfrac{1}{x^4}}{x+\dfrac{1}{x}}$. [Multiply numerator and denominator by x^4.]

10. $\frac{3ax}{4by} \times \frac{a^2-x^2}{c^2-x^2} \div \left(\frac{a^2+ax}{bc+bx} \times \frac{a-x}{c-x}\right)$.

11. $\left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right)^2 \div \frac{a-b}{a+b} \times \frac{a^3-b^3}{a^3+b^3}$.

12. $\frac{1-x^2}{1+y} \times \frac{1-y^2}{x+x^2} \div (1-x)$.

13. $\frac{a^2b^2+7ab}{4a^2-1} \div \frac{ab+7}{2a+1} \times \frac{1}{a}$.

14. $\dfrac{x^2-x-20}{x^2-25}\times\dfrac{x^2-x-2}{x^2+2x-8}\div\dfrac{x+1}{x^2+5x}.$

15. $\dfrac{4x^2+x-14}{6xy-14y}\times\dfrac{4x^2}{x^2-4}\times\dfrac{x-2}{4x-7}\times\dfrac{3x^2-x-14}{2x^2+4x}.$

16. $\dfrac{x^2+x-2}{x^2-x-20}\div\left(\dfrac{x^2+3x+2}{x^2-2x-15}\times\dfrac{x+3}{x^2}\div\dfrac{x^2+5x+4}{x^2-x}\right).$

17. $\dfrac{x^2-18x+80}{x^2-5x-50}\div\dfrac{x^2-15x+56}{x^2-6x-7}\div\dfrac{x-1}{x+5}.$

18. $\dfrac{6x^2-ax-2a^2}{ax-a^2}\cdot\dfrac{x-a}{9x^2-4a^2}\cdot\dfrac{3ax+2a^2}{2x+a}.$

19. $\dfrac{x^3+27y^3}{(x+3y)^3}\div\left(\dfrac{x}{y}\times\dfrac{x^2-3xy+9y^2}{(x+3y)^2}\right).$

20. $\dfrac{x^2-64}{x^2+24x+128}\cdot\dfrac{x^2+12x-64}{x^3-64}$
$$\div\left[\frac{(x-8)^2}{x^2+4x+16}\div\frac{x^2+10x+16}{x^2-64}\right]^2.$$

One Term Not Easily Factorable.

219. Sometimes in reducing fractions to lowest terms only one of the terms of the fraction can be factored by inspection. Then it is best to try the separate factors as divisors of the unfactored term.

Model B.—Reduce to lowest terms $\dfrac{x^2+2x-3}{3x^3+7x-10}.$

$$\frac{x^2+2x-3}{3x^3+7x-10}=\frac{(x+3)(x-1)}{3x^3+7x-10}.$$

Evidently $x+3$ cannot divide the denominator exactly because 3 is not an exact divisor of the straight product 10. Dividing by $x-1$ we get for the answer

$$\frac{x+3}{3x^2+3x+10}.$$

$$\begin{array}{l}
x-1 \\
\hline
3x^2+3x+10 \\
\hline
3x^3+7x-10 \\
3x^3-3x^2 \\
\hline
3x^2+7x \\
3x^2-3x \\
\hline
10x-10
\end{array}$$

EXERCISE XCIX.

Reduce to lowest terms:

1. $\dfrac{3a^2 - 6ab}{2a^2b - 4ab^2}$.

2. $\dfrac{a(2b^2 - 2ba)}{b(4b^2a - 9a^3)}$.

3. $\dfrac{3x^4 + 9x^3y + 6x^2y^2}{x^4 + x^3y - 2x^2y^2}$.

4. $\dfrac{x^2 + xy - 2y^2}{x^3 - y^3}$.

5. $\dfrac{27a + a^4}{18a - 6a^2 + 2a^3}$.

6. $\dfrac{x^2 - 5x + 4}{4x^2 + 9x - 13}$.

7. $\dfrac{2x^3 - 103x + 35}{x^2 - 13x + 42}$.

8. $\dfrac{3x^3 - 78x + 15}{2x^2 - 13x + 15}$.

9. $\dfrac{x^2 - 3x - 10}{7x^3 - 23x + 10}$.

10. $\dfrac{6x^2 + x - 12}{12x^3 - 23x + 6}$.

Test for Simple Factors.

220. It is sometimes easy to decide whether a very simple factor is contained without remainder in an algebraic expression without performing the operation of division. The test depends upon the axiom that if one factor of an expression is equal to zero, then the whole expression must be.

Model C.—To decide whether $x - 1$ is a factor of $x^3 + 5x^2 + x - 10$ it is only necessary to suppose that $x = 1$; in that case $x - 1 = 0$ and $x^3 + 5x^2 + x - 10 = 7 - 10 = -3$; hence $x - 1$ is not a factor, because when $x - 1 = 0$ the expression $x^3 + 5x^2 + x - 10$ is not.

In the same way $x + 1$ is not a factor, because, if $x = -1$, $x^3 + 5x^2 + x - 10$ becomes $-1 + 5 - 1 - 10 = -7$; and $x - 2$ is not a factor, because, if $x = 2$, $x^3 + 5x^2 + x - 10$ becomes $8 + 20 + 2 - 10 = 20$; but $x + 2$ IS a factor, because, if $x = -2$, $x^3 + 5x^2 + x - 10$ becomes $-8 + 20 - 2 - 10 = 0$.

EXERCISE C.

In the same way decide whether either or both of the binomials given with each of the following expressions are factors of that expression or not:

		Possible Factors.	
1.	x^3+2x^2-5x-6	$x+1$	$x-2$
2.	x^3-2x^2-5x+6	$x+3$	$x+2$
3.	$x^3-13x-12$	$x+1$	$x-1$
4.	x^3-7x+6	$x-2$	$x-1$
5.	x^3-3x^2+4	$x-2$	$x+2$
6.	$x^3-19x+30$	$x-2$	$x-3$
7.	$x^3-5x^2-2x+24$	$x-2$	$x+3$
8.	$x^4-11x^2+14x-24$	$x+1$	$x-2$
9.	$x^4+x^3-10303x^2+x-10302$	$x+1$	$x-1$
10.	$3x^4-7x^2-20$	$x+2$	$x-2$
11.	x^n-a^n	$x+a$	$x-a$
12.	x^n+a^n	$x+a$	$x-a$

221. Theoretically the same test would serve to decide whether $3x-2$ was a factor of $3x^4+13x^2-11x-18$; but practically it is easier, in this case, to perform the operation of long division than to substitute in the given expression $x=\frac{2}{3}$. The pupil must use his judgment as to which test he shall apply in each example.

EXERCISE CI.

Find the H. C. F. in each of the following examples:

1. $2x^3-3x^2-5x+6$; x^2-x-2.
2. $5x^3-21x^2+16$; x^2-x-12.
3. x^2-3x+2; $7x^5+13x^2-20$.
4. $3x^5+2x^4-47x-34$; x^2-4.
5. $3x^4+7x-62$; $x^2+29x-62$.

Neither Term Easily Factorable.

222.* In the following example it is extremely difficult to factor either the numerator or the denominator by inspection.

Model D.—Reduce to lowest terms $\dfrac{18x^3 + 13x - 14}{24x^3 + 2x^2 - 8}$.

We must try, then, in some other way to find a common divisor for these two expressions. The way we shall adopt is to change these expressions into others which have the same common divisors and at the same time are easier to factor.

Let us take as an abbreviation for the numerator the letter P, and for the denominator Q.

Q has a factor 2, which is NOT a factor of P and is therefore not a common factor. Divide Q by 2, and we get

$$12x^3 + x^2 - 4;$$

multiply this by 3, and we get

$$36x^3 + 3x^2 - 12.$$

Since we have not introduced or taken out common factors, any common divisor of P and Q is also a common divisor of P and $36x^3 + 3x^2 - 12$.

In the same way $2 \times P = 36x^3 + 26x - 28$.

Any number that will divide each of two quantities will also divide their sum or their difference. [This is merely one form of THE DISTRIBUTIVE LAW. We may illustrate it, but we cannot prove it, because it is one of those fundamental principles that have to be taken for a basis of all proofs.]

* See also §§ 229, 230.

So any number that is a divisor of P and Q is also a divisor of $36x^3 + 3x^2 - 12$ and of $36x^3 + 26x - 28$; and is therefore a divisor of THEIR DIFFERENCE, that is, of $3x^2 - 26x + 16$.

The advantage of this result lies in the fact that we have ELIMINATED, so to speak, the x^3 terms from our two expressions, and so obtained a QUADRATIC expression, which contains among its factors every common factor of P and Q. Factoring,

$$3x^2 - 26x + 16 \equiv (3x - 2)(x - 8).$$

We see that $x - 8$ cannot be a common factor, because 8 will not divide 14, one of the straight products. Dividing numerator and denominator by $3x - 2$, we get for the answer to the example:

$3x - 2$	$3x - 2$
$6x^2 + 4x + 7$	$8x^2 + 6x + 4$
$18x^3 + 13x - 14$	$24x^3 + 2x^2 - 8$
$18x^3 - 12x^2$	$24x^3 - 16x^2$
$12x^2 + 13x - 14$	$18x^2 - 8$
$12x^2 - 8x$	$18x^2 - 12x$
$21x - 14$	$12x - 8$
$21x - 14$	$12x - 8$
0	0

$$\frac{6x^2 + 4x + 7}{8x^2 + 6x + 4}$$

EXERCISE CII.

Reduce to lowest terms:

1. $\dfrac{3x^3 - 11x^2 + 18}{2x^3 - 23x + 15}$.

2. $\dfrac{3x^3 + 23x^2 - 50}{9x^3 - 19x + 10}$.

3. $\dfrac{12x^3 - 31x + 6}{8x^3 - 28x + 15}$.

4. $\dfrac{21x^3 - 92x^2 + 8}{49x^3 - 18x - 4}$.

5. $\dfrac{25x^6 - 60x^4 + 49}{25x^6 - 64x^2 + 21}$.

More Difficult Examples.

223. When we have for terms of our fraction expressions of degree higher than 3, this process will generally have to be carried a little further.

Model E.—Reduce $\dfrac{x^4 - 3x - 10}{2x^4 - 5x^3 + 8}$.

Here we destroy the HIGHEST terms as before. For convenience we may number the expressions and make a memorandum of how we get them, just as we do with equations:

(1) $x^4 - 3x - 10$	
(2) $2x^4 - 5x^3 + 8$	
(3) $2x^4 - 6x - 20$	(1) $\times$ 2
(4) $5x^3 - 6x - 28$	(3) $-$ (2)

This (4) is an expression of the third degree, which we cannot easily factor. We know, however, that it contains all the factors common to (1) and (2). We can obtain another such expression by destroying ("eliminating") the lowest terms.

(5) $4x^4 - 12x - 40$	(1) $\times$ 4
(6) $10x^4 - 25x^3 + 40$	(2) $\times$ 5
(7) $14x^4 - 25x^3 - 12x$	(5) $+$ (6)

This contains a factor x, and as no factor x occurs in either of the expressions we are investigating, we shall not affect our result if we cast it out.

(8) $14x^3 - 25x^2 - 12$	(7) $\div x$.

By the same reasoning as before, (8) contains all the factors that are common to (1) and (2). Therefore the H.C.F. of (4) and (8), if we could find it, would contain the H.C.F.

of (1) and (2).* We now apply ourselves to a new problem, —to find the H. C. F. of (4) and (8).

(7)	$5x^3 - 6x - 28$	
(8)	$14x^3 - 25x^2 - 12$	
(9)	$70x^3 - 84x - 392$	(4) $\times$ 14
(10)	$70x^3 - 125x^2 - 60$	(8) $\times$ 5
(11)	$125x^2 - 84x - 332$	(9) $-$ (10)

This expression (11), though only a quadratic, is still difficult to factor. We are aided by the fact that (1) and (2), the given numbers, have for straight products x^4 and $2x^4$, -10 and 8, and consequently if there is a common factor which is also a factor of (11), that factor may be $x+2$ or $x-2$. By trial we find the factors of (11) to be $(125x+166)(x-2)$. If there is a common factor, then, of (1) and (2), it must be $x-2$. Dividing:

$$\begin{array}{l}
x-2 \\
\hline
x^3+2x^2+4x+5 \\
\hline
x^4-3x-10 \\
x^4-2x^3 \\
\hline
2x^3-3x-10 \\
2x^3-4x^2 \\
\hline
4x^2-3x-10 \\
4x^2-8x \\
\hline
5x-10
\end{array}
\qquad
\begin{array}{l}
x-2 \\
\hline
2x^3-x^2-2x-4 \\
\hline
2x^4-5x^3+8 \\
2x^4-4x^3 \\
\hline
-x^3+8 \\
-x^3+2x^2 \\
\hline
-2x^2+8 \\
-2x^2+4x \\
\hline
-4x+8
\end{array}$$

Ans. $\frac{x^3+2x^2+4x+5}{2x^3-x^2-2x-4}$.

EXERCISE CIII.

Reduce to lowest terms:

1. $\frac{x^4+3x^3-27x+14}{x^4-15x+14}$.

2. $\frac{4x^4+11x^2+25}{4x^4-9x^2+30x-25}$.

3. $\frac{1+2x^2+x^3+2x^4}{1+3x^2+2x^3+3x^4}$.

4. $\frac{6x^3+x^2-5x-2}{6x^3+5x^2-3x-2}$.

5. $\frac{9x^4-6x^3-343}{9x^4-49x^2+6x+14}$.

* So far as we have yet proved, it might contain other factors. See further.

Theory of H. C. F. by Elimination.

224. If P and Q be the numerator and denominator, respectively, of a fraction which we are to reduce to lowest terms, the process used above may be carried out in symbols, and some general conclusions can be drawn from the formulæ thus obtained.

To destroy (or eliminate) the terms of highest degree we multiply P and Q, respectively, by some suitable multipliers, say a and b, and subtract. The result, which we may call X, is an expression of lower degree than P and Q.

$$(1)\ X = aP - bQ.$$

Then destroying the terms of lowest degree, and letting h and k stand for our two multipliers:

$$(2)\ Y = \frac{hP - kQ}{x}.$$

Now it is evident from (1) and (2) that **any common factor of P and Q is** a factor of X, and also a factor of Y, that is, **a common factor of X and Y.** Let us take these two equations and find an expression for P and Q, to see if some other conclusion can be drawn.

(3)	$xY = hP - kQ$	(2) $\times x$
(4)	$hX = ahP - bhQ$	(1) $\times h$
(5)	$axY = ahP - akQ$	(3) $\times a$
(6)	$hX - axY = akQ - bhQ$	(4) $-$ (5)
(7)	$\dfrac{hX - axY}{ak - bh} = Q$	(6) $\div (ak - bh)$

Here we have found an expression for Q by eliminating P, and from that value of Q it is evident that **any common factor of X and Y is also a factor of Q.** In the same way it would be found that

$$(8)\ P = \frac{kX - bxY}{ak - bh};$$

and it is established that **any common factor of X and Y is also a factor of P.**

Therefore, since X and Y have for common factors all the common factors of P and Q, and no others, **the H. C. F. of X and Y is the same as that of P and Q**; and the elimination may be continued, taking each pair of results as a new problem, until the H. C. F. appears in two identical expressions.

Model F.

①	Let $P = 4x^4 + 26x^3 + 41x^2 - 2x - 24$	
②	$Q = 3x^4 + 20x^3 + 32x^2 - 8x - 32$	
③	$12x^4 + 78x^3 + 123x^2 - 6x - 72$	① $\times$ 3
④	$12x^4 + 80x^3 + 128x^2 - 32x - 128$	② $\times$ 4
⑤	$2x^3 + 5x^2 - 26x - 56$	$X = 4Q - 3P$
⑥	$16x^4 + 104x^3 + 164x^2 - 8x - 96$	① $\times$ 4
⑦	$9x^4 + 60x^3 + 96x^2 - 24x - 96$	② $\times$ 3
⑧	$7x^4 + 44x^3 + 68x^2 + 16x$	⑥ $-$ ⑦
⑨	$7x^3 + 44x^2 + 68x + 16$	⑧ $\div x$ $\quad Y = \dfrac{4P - 3Q}{x}$

Now since the H. C. F. of X and Y is the same as the H. C. F. of P and Q, we may start on a new problem, namely, to find the H. C. F. of $2x^3 + 5x^2 - 26x - 56$ and $7x^3 + 44x^2 + 68x + 16$, which we may call P and Q respectively.

⑩	Let $P = 2x^3 + 5x^2 - 26x - 56$	
⑪	$Q = 7x^3 + 44x^2 + 68x + 16$	
⑫	$14x^3 + 35x^2 - 182x - 392$	⑩ $\times$ 7
⑬	$14x^3 + 88x^2 + 136x + 32$	⑪ $\times$ 2
⑭	$53x^2 + 318x + 424$	$X = 2Q - 7P$
⑮	$4x^3 + 10x^2 - 52x - 112$	⑩ $\times$ 2
⑯	$49x^2 + 308x^2 + 476x + 112$	⑪ $\times$ 7
⑰	$53x^3 + 318x^2 + 424x$	⑮ + ⑯
⑱	$53x^2 + 318x + 424$	$Y = \dfrac{2P + 7Q}{x}$

Now since X and Y are the same expression, their H. C. F. is that expression, and that is the H. C. F. of P and Q, — and that is the H. C. F. sought.

EXERCISE CIV.

Reduce wherever possible:

1. $\dfrac{2x^5 - 11x^2 - 9}{4x^5 + 11x^4 + 81}$.

2. $\dfrac{2x^4 - 2x^3 + x^2 + 3x - 6}{4x^4 - 2x^3 + 3x - 9}$.

3. $\dfrac{x^4 - x^3 + 2x^2 + x + 3}{x^4 + 2x^3 - x - 2}$.

4. $\dfrac{6x^4 + x^3 - x}{4x^3 - 18x^2 + 19x - 3}$.

5. $\dfrac{3x^4 - 3x^3 - 2x^2 - x - 1}{6x^4 - 3x^3 - x^2 - x - 1}$.

6. $\dfrac{x^4 - 10x^2 + 9}{x^4 + 10x^3 + 20x^2 - 10x - 21}$.

7. $\dfrac{x^5 + 5x^4 - x^2 - 5x}{x^4 + 3x^3 - x - 3}$.

8. $\dfrac{x^5 - 16x - 32}{x^5 - 3x^3 - 6x^2 + 16x - 8}$.

9. $\dfrac{96x^4 + 8x^3 - 2x}{32x^3 - 24x^2 - 8x + 3}$.

10. $\dfrac{x^4 - x^2 - 2x + 2}{2x^3 - x - 1}$.

Another Application of H. C. F.

225. We have seen that an algebraic equation with one unknown letter can have no answers but those which make the several factors equal to zero. If there are two such equations, then, which can be satisfied by the same value of x, it follows that the two equations must have a common factor.

Model G.—There is a number which will satisfy each of the two equations

$$16x^4 - 8x + 3 = 0$$
$$64x^3 = 8$$

To find what that answer is, we first find the H. C. F. of the two expressions $16x^4 - 8x + 3$ and $64x^3 - 8$.

① $16x^4 - 8x + 3 = 0$
② $64x^3 - 8 = 0$
③ $8x^3 - 1$ ② $\div 8$

(4) $24x^3 - 3$	(3) $\times 3$
(5) $16x^4 + 24x^3 - 8x$	(1) + (4)
* (6) $2x^3 + 3x^2 - 1$	(5) $\div 8x$
(7) $16x^4 - 2x$	(3) $\times 2x$
(8) $6x - 3$	(7) − (1)
* (9) $2x - 1$	(8) $\div 3$

If there is an H. C. F. it must be $2x - 1$; we find on trial that this is contained in $2x^3 + 3x^2 - 1$ exactly $x^2 + 2x + 1$ times; hence $2x - 1$ is a factor of both equations (1) and (2). That is, each equation will be satisfied if $2x - 1 = 0$; hence $x = \frac{1}{2}$ is a root of each equation.

EXERCISE CV.

In each of the following examples there is one number that will satisfy both equations. Find that number.

1. $x^3 - 7x^2 + 16x - 12 = 0$; $3x^2 - 14x + 16 = 0$.
2. $x^3 - 3x^2 - 9x + 27 = 0$; $x^2 - 2x - 3 = 0$.
3. $2x^3 + 5x^2 - 6x - 9 = 0$; $3x^3 + 7x^2 - 11x - 15 = 0$.
4. $x^3 + 19x^2 - 22x - 40 = 0$; $x^3 + 11x^2 - 6x - 40 = 0$.
5. $4x^3 - 21x + 10 = 0$; $2x^3 + 9x^2 - 25 = 0$.
6. $x^4 - 11x^2 + 18x - 8 = 0$; $4x^3 - 22x + 18 = 0$.
7. $x^4 - 2x^3 - x^2 - 4x + 12 = 0$; $2x^3 - 3x^2 - x - 2 = 0$.
8. $x^4 - 7x^3 + 13x^2 + 3x - 18 = 0$; $4x^3 - 21x^2 + 26x + 3 = 0$.
9. $x^4 - 4x^3 - 6x^2 + 36x - 27 = 0$; $x^3 - 3x^2 - 3x + 9 = 0$.
10. $x^4 + 13x^3 + 33x^2 + 31x + 10 = 0$; $4x^3 + 39x^2 + 66x + 31 = 0$.

H. C. F. of Three Expressions.

226. Model H.—Find the H. C. F. of

$8x^4 + 3x^3 - 64x^2 + 9$; $9x^4 - 64x^2 + 3x + 8$; and $x^4 + x^3 - 8x^2 - 5x + 3$.

	(1) $8x^4 + 3x^3 - 64x^2 + 9$	
	(2) $x^4 + x^3 - 8x^2 - 5x + 3$	
	(3) $8x^4 + 8x^3 - 64x^2 - 40x + 24$	(2) $\times$ 8
	(4) $5x^3 - 40x + 15$	(3) $-$ (1)
*	(5) $x^3 - 8x + 3$	(4) $\div$ 5
	(6) $3x^4 + 3x^3 - 24x^2 - 15x + 9$	(2) $\times$ 3
	(7) $5x^4 - 40x^2 + 15x$	(1) $-$ (6)
*	(8) $x^3 - 8x + 3$	(7) $\div$ $5x$

It is now clear that $x^3 - 8x + 3$ is the H. C. F. of two of the given expressions; it contains, then, all factors common to those two. The H. C. F. of this result and the remaining one of the three given expressions will contain all the factors common to the three; so we seek the H. C. F. of

$$x^3 - 8x + 3 \quad \text{and} \quad 9x^4 - 64x^2 + 3x + 8.$$

	(1) $x^3 - 8x + 3$	
	(2) $9x^4 - 64x^2 + 3x + 8$	
	(3) $9x^4 - 72x^2 + 27x$	(1) $\times$ $9x$
	(4) $8x^2 - 24x + 8$	(2) $-$ (3)
*	(5) $x^2 - 3x + 1$	(4) $\div$ 8

The quadratic expression $x^2 - 3x + 1$ is prime; hence if there is an H. C. F., this must be it. Dividing, we find that

$$\frac{x^3 - 8x + 3}{x^2 - 3x + 1} \equiv x + 3; \quad \frac{9x^4 - 64x^2 + 3x + 8}{x^2 - 3x + 1} \equiv 9x^2 + 27x + 8.$$

Hence $x^2 - 3x + 1$ is the H. C. F. required.

EXERCISE CVI.

Find the H. C. F. of:

1. $x^3 - x - 6$; $4x^4 - x^2 - 18x + 9$; $2x^4 + 3x^3 + x^2 - 9x - 9$.
2. $2x^3 - 9x^2 + 13x - 6$; $4x^3 - 20x^2 + 31x - 15$; $2x^3 - 11x^2 + 19x - 10$.
3. $9x^4 - 10x^2 + 1$; $21x^4 + 10x^3 - 20x^2 - 10x - 1$; $21x^4 - 4x^3 - 22x^2 + 4x + 1$.
4. $6x - 11x^2 + 6x^3 - 1$; $19x^2 + 1 - 4x(3x^2 + 2)$; $3x(3 + 8x^2) - (1 + 26x^2)$.
5. $x^3 - 7x - 6$; $2x^3 - 7x^2 + 9$; $2x^3 - 5x^2 - 9x + 18$.

H. C. F. by Long Division.

227. In Arithmetic the H. C. F. of any two numbers is always found by dividing the larger by the smaller, the divisor by the remainder, and so on, continuing the operation until there is no remainder.

Model I.—To find the H. C. F. of 62651 and 18377

$$
\begin{array}{l}
18377)62651(3 = q_1 \\
\qquad\quad \underline{55131} \\
\quad r_1 = 7520)18377(2 = q_2 \\
\qquad\qquad\qquad \underline{15040} \\
\qquad\qquad r_2 = 3337)7520(2 = q_3 \\
\qquad\qquad\qquad\qquad\quad \underline{6674} \\
\qquad\qquad\qquad r_3 = 846)3337(3 = q_4 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{2538} \\
\qquad\qquad\qquad\qquad r_4 = 799)846(1 = q_5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{799} \\
\qquad\qquad\qquad\qquad\qquad r_5 = 47)799(17 = q_6 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{47} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 329 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{329} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

A much more compact arrangement of the work is here shown. [The successive quotients are lettered $q_1, q_2, q_3, \ldots$ and the successive remainders $r_1, r_2, r_3, \ldots$]

$q_2 = 2$	$b = 18377$	$62651 = a$	$3 = q_1$
	15040	55131	
$q_4 = 3$	$r_2 = 3337$	$7520 = r_1$	$2 = q_3$
	2538	6674	
$q_6 = 17$	$r_4 = 799$	$846 = r_3$	$1 = q_5$
	47	799	
	$r_6 = 329$	$47 = r_5$	
	329		
	0		

228. The theory of this method is substantially the same as that of the method by elimination of highest and lowest terms. If a and b are the dividend and divisor, and q and r the quotient and remainder, respectively, in any example in division, then

$$(1)\quad a = bq + r.$$

From this we see that a common factor of b and r will be a distributive factor of $bq + r$ and therefore a factor of a; that is, **A common factor of b and r is also a common factor of a and b.**

Again, by subtracting bq from each member of (1),

$$(2)\quad a - bq = r.$$

Hence a common factor of a and b is a distributive factor of $a - bq$, and is therefore a factor of r; that is, **A common factor of a and b is also a common factor of b and r.**

These two conclusions establish the fact that b and r have the same common factors as a and b, and no others; in other words, instead of finding the H. C. F. of a and b, we may find the H. C. F. of b and r.

In Model I, therefore,

the H. C. F. of a and b is the same as the H. C. F. of r_1 and b;

the H. C. F. of r_1 and b is the same as the H. C. F. of r_2 and r_1;

the H. C. F. of r_2 and r_1 is the same as the H. C. F. of r_3 and r_2;

the H. C. F. of r_3 and r_2 is the same as the H. C. F. of r_4 and r_3;

the H. C. F. of r_4 and r_3 is the same as the H. C. F. of r_5 and r_4.

Since r_5 is contained in r_4 without remainder, the H. C. F. of r_5 and r_4 is r_5; and r_5 is therefore the H. C. F. of a and b.

229. This method, if applied to algebraic expressions, very often requires important modifications.

Model J.—Find the H. C. F. of $7x^4 - 3x^3 - 11x - 66$ and $3x^4 + 2x^2 - 17x - 22$:

$$\begin{array}{r|l|l|l}
q_1 = 2 & a = 7x^4 - 3x^3 - 11x - 66 & 3x^4 + 2x^2 - 17x - 22 = b & 3 = q_2 \\
 & \qquad 6x^4 + 4x^2 - 34x - 44 & 3x^4 - 9x^3 - 12x^2 + 69x - 66 & \\
\hline
 & r_1 = x^4 - 3x^3 - 4x^2 + 23x - 22 & 9x^3 + 14x^2 - 86x + 44 = r_2 &
\end{array}$$

Here we find that while r_1 and r_2 have the same H. C. F. as a and b, their quotient will contain fractions whichever is used as divisor. Going back to first principles, we may say that since r_2 does not contain 9 as a distributive factor, we may introduce it as a distributive factor in r_1; then $9r_1$ and r_2 will have the same H. C. F. as r_1 and r_2. Some such device as this may have to be used several times in one example, or in rare cases may not be needed at all.

$q_1 = 2$	$a = 7x^4 - 3x^3 - 11x - 66$	$3x^4 + 2x^2 - 17x - 22 = b$	$3 = q_2$
	$6x^4 + 4x^2 - 34x - 44$	$3x^4 - 9x^3 - 12x^2 + 69x - 66$	
	$r_1 = x^4 - 3x^3 - 4x^2 + 23x - 22$	$9x^3 + 14x^2 - 86x + 44 = r_2$	
	9		
$q_3 = x$	$9x^4 - 27x^3 - 36x^2 + 207x - 198$		
	$9x^4 + 14x^3 - 86x^2 + 44x$		
	$r_3 = -41x^3 + 50x^2 + 163x - 198$		
	-9		
$q_4 = 41$	$369x^3 - 450x^2 - 1467x + 1782$		
	$369x^3 + 574x^2 - 3526x + 1804$		
	$r_4 = -1024x^2 + 2059x - 22$		
	$1024x^2 - 2059x + 22$		

Here again it would be profitable to depart from routine: rather than multiply r_2 by 1024, it would be easier to factor $1024x^2 - 2059x + 22$. We find, in fact, that it is the product of $x - 2$ and $1024x - 11$; and only the first of these will divide r_2; hence $x - 2$ is the H. C. F.

230. This method is somewhat more perplexing at times than the method of elimination of highest and lowest terms; and the following cautions must be carefully observed:

I. Remove distributive factors before beginning operations; if they are common to a and b, restore them at the end of your work.

II. When you find that any remainder will not serve as divisor, carefully look for distributive factors, and if there are any, remove them; THEN introduce whatever factors are necessary into the last divisor.

III. Continue to use the same divisor as long as possible.

IV. When you get a quadratic remainder, factor it.

All the examples given under the other method will serve for practice in this.

CHAPTER IX.

LEAST COMMON MULTIPLE; SUMMATION OF FRACTIONS.

231. If one expression is a factor of another, the second may be called a MULTIPLE of the first. Thus $x^4 - 16$ is a multiple of $x^2 + 4$, $x + 2$, $x - 2$, $x^2 - 4$, $x^3 - 2x^2 + 4x - 8$, and $x^3 + 2x^2 + 4x + 8$.

Again, $x^4 - 16$ is a **common** multiple of $x^3 - 2x^2 + 4x - 8$ and $x^3 + 2x^2 + 4x + 8$, just as 192 is a common multiple of 48 and 32.

232. The **lowest common multiple** of two or more expressions is the expression of lowest degree in which those expressions may be found as factors.

Thus $x^2 - 4$ and $x^2 + 4$ have $x^4 - 16$ for their lowest common multiple; just as 9 and 16 have 144 for their least common multiple in arithmetic; $x^2 + 4$ and $x + 2$ have for their **lowest** common multiple $x^3 + 2x^2 + 4x + 8$, although $x^4 - 16$ is a common multiple; and $x^2 - 5x + 6$ and $x^2 - 4$ have $x^3 - 3x^2 - 4x + 12$ for their lowest common multiple.

EXERCISE CVII.

In the following table the expression in the third column is the L. C. M. of the corresponding expressions in the first two; find out by what each of those first two expressions must be multiplied to give the L. C. M.

1. x^2-3x+2	x^2-5x+4	$x^3-7x^2+14x-8$
2. $x^2-2x-15$	x^2-25	$x^3+3x^2-25x-75$
3. x^2+x-20	$x^2+10x+25$	$x^3+6x^2-15x-100$
4. x^3-27	x^3+2x^2+6x-9	$x^4-x^3-27x+27$
5. $x^2+8x+12$	x^2+5x-6	$x^3+7x^2+4x-12$
6. $2x^2+x-6$	$2x^2-5x+3$	$2x^3-x^2-7x+6$
7. x^2y-xy^2	$x^4y^2-x^2y^4$	$x^4y^2-x^2y^4$
8. $3x^2-x-10$	$2x^2-8$	$6x^3+22x^2-28x-80$
9. $5x^2-9x-2$	$25x^2-1$	$25x^3-50x^2-x+2$
10. $10x^2-9x-9$	$50x^3-18x$	$100x^4-120x^3-36x^2+54x$

Very Simple Denominators.

233. In adding fractions it is most convenient to use the Lowest Common Multiple of the denominators.

Model A.—Reduce

$$\frac{x-5}{3}-\frac{2x-13}{6}+\frac{2x}{8}-\frac{5+2x}{12}$$

The product of all the denominators would be a common multiple, but it would be a very large number; the least common multiple is 24. We can reduce each of these fractions then to 24ths.

The first fraction has to have its denominator multiplied by 8 to give 24; then the numerator must be multiplied also by 8, in order not to change the value of the fraction; similarly for the other fractions; so we get

$$\frac{8x-40}{24}-\frac{8x-52}{24}+\frac{6x}{24}-\frac{10+4x}{24}$$

which reduces to

$$\frac{2x-10}{24}$$

and this again reduces to

$$\frac{x-5}{12}$$

EXERCISE CVIII.

In the same way reduce:

1. $\dfrac{x-5}{3}+\dfrac{2x-3}{5}-\dfrac{5-2x}{30}$. 2. $\dfrac{2x-1}{6}-\dfrac{3x-2}{10}+\dfrac{2-5x}{15}$.

3. $\dfrac{x^2-3}{4}-\dfrac{2x^2-x-5}{6}-\dfrac{5-2x-3x^2}{12}$

4. $\dfrac{3a-b}{8}-\dfrac{2b-7a}{10}-\dfrac{3a}{20}$.

5. $\dfrac{x^3-27}{5}-\dfrac{2x^2-5+x^3}{3}+\dfrac{-x^3-8}{21}$.

6. $\dfrac{(x-2)(x-3)}{7}+\dfrac{(2-x)(3-x)}{21}-\dfrac{(x-2)(3-x)}{14}$.

7. $\dfrac{(3x-5)(2x+3)}{2}+\dfrac{(7x-1)(2x-1)}{6}-\dfrac{(x-3)(3-x)}{18}$

8. $\left(\dfrac{3x+5y}{2}\right)\left(\dfrac{3x-5y}{3}\right)-\left(2x-3y\right)\left(\dfrac{3x-2y}{6}\right)$.

9. $\left(\dfrac{17x-29}{5}\right)\left(\dfrac{x-3}{4}\right)+\left(\dfrac{29-17x}{2}\right)\left(\dfrac{x+2}{10}\right)$.

10. $\left(\dfrac{x^2-3x+9}{18}\right)\left(\dfrac{x+3}{2}\right)+\left(3-x\right)\left(\dfrac{x^2+3x+9}{36}\right)$.

11. $\dfrac{x+1}{3}\quad\dfrac{2x-1}{6}+\dfrac{x^2-4}{x-3}$. 12. $\dfrac{5x^2}{x-4}-\dfrac{x}{4}-\dfrac{1-x}{10}$.

13. $\dfrac{3x^2+7}{x+5}+\dfrac{2x^2-3}{3x+15}-\dfrac{x+1}{9}$. 14. $\dfrac{x^2-y^2}{x+y}+\dfrac{x^2}{4x+4y}+\dfrac{y}{2}$.

15. $-\dfrac{x+1}{x-3}-\dfrac{5}{9}-\dfrac{2x+3}{2x-6}$. 16. $\dfrac{x^2+7}{2x-1}-\dfrac{x}{4x-2}+\dfrac{-3x-6}{6x-3}$.

17. $\dfrac{(x+2)(x+3)}{x+3}+\dfrac{(x+2)(x-1)}{2x+6}-\dfrac{x-1}{12}+\dfrac{(x+2)(x-2)}{3x+9}.$

18. $\dfrac{3x^2}{x-7}+\dfrac{x^2-3}{49-7x}+\dfrac{x-3}{14}.$ 19. $\dfrac{2}{3x-2}-\dfrac{5}{3x-2}+\dfrac{4}{12-18x}.$

20. $\dfrac{5x}{x-y}+\dfrac{6x}{7x-7y}-\dfrac{7x}{8x-8y}+\dfrac{3x^2}{x^2-y^2}.$

Factoring for the L. C. M.

234. The L.C.M. of several algebraic expressions may often be found by factoring.

Model B.—Find the L. C. M. of

$x^2-9;\ x^3-27;\ x^2-4x+3;\ x^4+9x^2+81;\ x^3+27.$

Here the L.C.M. must contain all the factors of any one of the expressions; we find the factors to be:

① $x^2-9 \equiv (x+3)(x-3)$
② $x^3-27 \equiv (x-3)(x^2+3x+9)$
③ $x^2-4x+3 \equiv (x-3)(x-1)$
④ $x^4+9x^2+81 \equiv (x^2+3x+9)(x^2-3x+9)$
⑤ $x^3+27 \equiv (x+3)(x^2-3x+9)$

The L.C.M. contains the factors of ①; then to make it contain the factors of ②, x^2+3x+9 must be included, and so on; including in the L.C.M. successively the factors of each number that are not already in it.

Starting with the factors of ①, the L.C.M. includes as factors

$$(x+3)(x-3)$$

Then to contain ② the L.C.M. must also include (x^2+3x+9)
Then to contain ③ the L.C.M. must also include $(x-1)$
Then to contain ④ the L.C.M. must also include (x^2-3x+9)
Then to contain ⑤ the L.C.M. needs no more factors.

EXERCISE CIX.

Find the L.C.M. of:

1. $x^2 - 121$; $x^2 - 2x - 99$; $5x^3 - 55x^2$.
2. $27x^3 - 8$; $36x^2 - 16$; $6x^2 + 11x - 10$; $9x^2 + 9x - 10$.
3. $x^3 - 8$; $x^4 + 4x^2 + 16$; $x^3 + 4x^2 - 8x - 8$.
4. $a^3 + b^3$; $a - b$; $a^4 + a^2b^2 + b^4$.
5. $x^2 - x(a + b) + ab$; $x^2 - a^2$; $(x + a)(x - b)^2$.
6. $x^3 + 3x^2$; $x^3 - 9x$; $x^3 + 27$; $(x + 3)^2 - 9x$.
7. $x^5 - y^5$; $x^2 - y^2$; $x + y$.
8. $x^{15} - y^{15}$; $x^3 - y^3$; $x^5 - y^5$; $x - y$; $x^2 + xy + y^2$.
9. $6x^2 - 18x + 12$; $4x^2 - 16x + 12$; $2x^2 - 10x + 12$.
10. $x^2 - (y + z)^2$; $y^2 - (x + z)^2$; $z^2 - (x + y)^2$.

More Complicated Denominators.

235. Model C.—Simplify the expression:

$$\frac{3x-1}{x^3-27} - \frac{3}{x^4+9x^2+81} + \frac{3x+1}{x^3+27} - \frac{3}{x^2-9}.$$

The L. C. M. is

$(x - 3)(x^2 + 3x + 9)(x + 3)(x^2 - 3x + 9) \equiv x^6 - 729$.

The first denominator must be multiplied by

$$(x + 3)(x^2 - 3x + 9)$$

to give the L. C. M.; then the numerator must be multiplied by the same expression:

$$\frac{3x-1}{x^3-27} \equiv \frac{3x^4 - x^3 - 81x + 27}{x^6 - 729}.$$

Similarly for the other fractions. The entire expression thus becomes

$$\frac{3x^4-x^3-81x+27}{x^6-729} - \frac{3x^2-27}{x^6-729} + \frac{3x^4+x^3-81x-27}{x^6-729} - \frac{3x^4-27x^2-243}{x^6-729}$$

$$\equiv \frac{3x^4+24x^2-162x+270}{x^6-729}. \quad \textit{Ans.}$$

EXERCISE CX.

Simplify the expressions:

1. $\dfrac{3x}{4x^2-9}+\dfrac{7}{2x+3}-\dfrac{2}{2x-3}$.

2. $\dfrac{2+x^2}{x^2-4}+\dfrac{x-3}{3x-6}-\dfrac{1-x}{8x+16}$.

3. $\dfrac{x}{x^3-27}-\dfrac{3}{x^2-5x+6}+\dfrac{4}{3x^2+9x+27}$.

4. $\dfrac{x+1}{2x^2-18}-\dfrac{1-x}{x^2-4x+3}+\dfrac{x-2}{x^2-1}-\dfrac{2-x}{x^2+2x-3}$.

5. $\dfrac{2}{3}+\dfrac{x^2-1}{x^2-7x+10}-\dfrac{x^2+x+1}{x^2-3x-10}-\dfrac{2x-3}{3x-15}$.

6. $\dfrac{a^2+b^2}{a^3-b^3}+\dfrac{a}{a^2+ab+b^2}-\dfrac{b}{a^2-ab}+\dfrac{a+b}{ab-b^2}$.

7. $\dfrac{(x^2+y^2)^2}{x^4+x^2y^2+y^4}-\dfrac{x^2+y^2}{x^2-xy+y^2}-\dfrac{x^2}{x^2+xy+y^2}$.

8. $\dfrac{3x+2}{x^2-3x-28}-\dfrac{2x-3}{x^2-16}+\dfrac{x+2}{x^2-11x+28}$.

9. $\dfrac{2x^2+3}{4x^2-36}+\dfrac{7x}{4x-12}-\dfrac{2x}{5x+15}-1$.

10. $3+\dfrac{x^2-3}{x^2-9}-\dfrac{x-3}{x+3}$.

Fractional Equations.

236. An expression which forms one entire member of an equation may have its value changed, without destroying the equation, provided the other member of the equation also has its value changed in the same way and to the same extent; but we must be careful not to change the value of an expression which is not a member of an equation.

Thus in the equation

$$\frac{x-5}{2}+\frac{3x-2}{7}=\frac{x+7}{14}-\frac{2x-3}{4}+\frac{1}{28}$$

we may "clear of fractions" by multiplying by 28; but in the expression

$$\frac{x-5}{2}+\frac{3x-2}{7}$$

we can NOT clear of fractions; we have no right to multiply by 14 or by anything else; we may reduce to one fraction

$$\frac{7x-5}{14}+\frac{6x-4}{14}\equiv\frac{13x-9}{14}$$

but we cannot lose sight of the denominator.

EXERCISE CXI.

Simplify the following equations:

1. $\frac{x-1}{2}+\frac{3x+2}{6}+\frac{6}{x-3}=\frac{x^2-x}{x-3}$.

2. $\frac{4x+7}{5}+\frac{3x-4}{15}=\frac{20x-3}{15}$

3. $\frac{4x-3y}{7}+\frac{3x+7y}{14}-\frac{5x-2y}{21}+\frac{9x+2y}{42}=\frac{3y}{14}+\frac{2(x+1)}{3}$.

4. $\frac{1}{x-6}+\frac{1}{x+5}=\frac{8-x}{(x-6)(x-5)}$.

5. $\frac{1}{1-x}-\frac{2}{1-x^2}=\frac{1}{2x-5}$.

6. $\frac{16x-x^2}{x^2-4}-\frac{3+2x}{x-2}-\frac{2-3x}{2+x}=\frac{1}{8}$.

7. $\frac{1}{x^2+9x+20}+\frac{1}{x^2+12x+35}=\frac{1}{x^2+11x+28}$.

8. $\dfrac{1}{x^2-13x+42}+\dfrac{1}{x^2-15x+54}=\dfrac{3}{x^2-16x+63}.$

9. $\dfrac{1}{x^2+7x-44}+\dfrac{1}{x^2-2x-143}=\dfrac{1}{x^2-17x+68}.$

10. $\dfrac{1}{x^2+3x+2}+\dfrac{2x}{x^2+4x+3}+\dfrac{1}{x^2+5x+6}=\dfrac{x-1}{x+3}.$

The Three Principal Signs.

237. The entire numerator of an algebraic fraction represents some number which may be $+$ or $-$ according as one set of values or some other set is assigned for the letters appearing in it; so also the denominator of an algebraic fraction may be $+$ or $-$; and the quotient represented by the fraction would have its sign $+$ or $-$ according as the entire numerator and the entire denominator had signs alike or unlike.

In the expression $x-\dfrac{a-q}{b-q}$ the values $a=35$, $b=8$, $q=5$ give to the fraction the value $\frac{30}{3}=10$, so that the value of the expression is $x-10$.

The fraction is unchanged if it be written $\dfrac{q-a}{q-b}$, because in this case we have changed the signs of numerator and denominator both.

If, however, we write the fraction $\dfrac{q-a}{b-q}$, its value becomes $\dfrac{-30}{3}=-10$; and if we then wish the value of the expression to be unaltered, we must change the sign of the term in which the fraction appears. Thus $x-\dfrac{q-a}{b-q}$ would be $x-(-10)\equiv x+10$; to keep the entire expression unchanged in value we must write it $x+\dfrac{q-a}{b-q}$.

So also if we write the fraction $\frac{a-q}{q-b}$, then we must write the expression $x + \frac{a-q}{q-b}$.

238. Whenever an algebraic fraction appears as a term in an expression, it is convenient to recognize as the **THREE PRINCIPAL SIGNS** of the fraction:

I. The Sign of the Whole Numerator;

II. The Sign of the Whole Denominator;

III. The Sign before the Fraction;

and any two of the three principal signs of a fraction may be reversed without altering the value of the expression in which the fraction appears.

Model D.—In the expression

$$\frac{3}{x-1} - \frac{4}{1-x^2} + \frac{2}{1+x}$$

it is desirable that the second denominator should be written $x^2 - 1$ so as to bring out more clearly the fact that the first and third denominators are factors of it. The expression may be written

$$\frac{3}{x-1} + \frac{4}{x^2-1} + \frac{2}{x+1}$$

without altering its value.

Model E

$$\frac{x+7}{(x-3)(x-5)} - \frac{2x-1}{(x-2)(3-x)} + \frac{2-3x}{(2-x)(5-x)}$$

may be written

$$\frac{x+7}{(x-3)(x-5)} + \frac{2x-1}{(x-2)(x-3)} - \frac{3x-2}{(x-2)(x-5)}$$

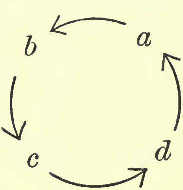
without changing its value; as follows:

———in the first fraction there is no change;
———in the second fraction the sign of one factor of the denominator is changed; this changes the sign of the whole denominator, which is one of the three principal signs; so the sign of the whole fraction is also changed;
———in the third fraction we change the sign of two factors, which leaves the sign of the whole denominator unchanged; the sign of the numerator is changed, so the sign of the whole fraction must also be changed.

Rearrangements.

239. The altered form of the above expressions will be found much more convenient in the process of uniting the three fractions into one. The principle of arrangement in this case is to have the x-term in each binomial come first, so that one may apply mentally the process of cross-multiplication.

240. There is another principle of arrangement which it is sometimes expedient to follow. It is known as **cyclic order**, and is of importance to all students of Algebra.



According to this principle, if we had four letters a, b, c, d, we should make b follow a, c follow b, d follow c, a follow d, b follow a, and so on; the several letters taking turns and beginning again at the first, as if they were arranged in a circle. Thus the expression

$$(a-b)(b-c)-(b-c)(a-c)+(b-a)(c-b)$$

would be written, according to this principle, thus:

$$(a-b)(b-c)+(b-c)(c-a)+(a-b)(b-c)$$

without change of value.

EXERCISE CXII.

Simplify the following expressions:

*1. $\dfrac{1}{x-2}+\dfrac{2}{x+2}+\dfrac{3}{2-x}$.

2. $\dfrac{x}{3x+1}+\dfrac{1}{9x^2-1}+\dfrac{x+1}{1-3x}$.

3. $\dfrac{3}{1-4x^2}+\dfrac{x}{2x-1}+\dfrac{1-x}{2x+1}$.

4. $\dfrac{1}{(x-2)(x-3)}+\dfrac{1}{(x+2)(x-3)}+\dfrac{2}{4-x^2}$.

5. $\dfrac{2}{x^2-7x+12}-\dfrac{3}{12-x-x^2}+\dfrac{5}{48-3x^2}$.

6. $\dfrac{c}{(a-b)(a-c)}+\dfrac{a}{(c-b)(c-a)}+\dfrac{c}{(b-a)(b-c)}$.

7. $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(b-c)(a-c)}+\dfrac{1}{(a-c)(b-a)}$.

8. $\dfrac{a+b}{(b-c)(a-c)}+\dfrac{b+c}{(b-a)(c-a)}$.

9. $\dfrac{a}{(a-b)(c-a)}+\dfrac{b}{(c-b)(c-a)}+\dfrac{b}{(b-a)(b-c)}$.

10. $\dfrac{a^2}{(b-a)(b-c)}+\dfrac{b^2}{(a-b)(a-c)}$.

11. $\dfrac{a}{(b-c)(c-a)}-\dfrac{2b}{(a-b)(a-c)}-\dfrac{2b}{(b-c)(b-a)}$.

12. $\dfrac{x}{(x-5)(y-x)}+\dfrac{5}{(y-5)(y-x)}+\dfrac{5}{(5-x)(5-y)}$.

13. $\dfrac{x}{(5-y)(y-x)}-\dfrac{10}{(x-5)(x-y)}-\dfrac{10}{(5-y)(5-x)}$.

* Find the shortest way of doing this example.

14. $\dfrac{x+5}{(5-y)(x-y)} - \dfrac{5+y}{(5-x)(y-x)}$.

15. $\dfrac{x^2}{(y-x)(y-7)} + \dfrac{y^2}{(x-y)(x-7)}$.

If $y = 2$ in the following expressions, what value must x have in each case to make the whole expression equal to $\frac{2}{5}$?

*16. $\dfrac{1}{x+y} - \dfrac{1}{x-y} + \dfrac{2x}{x^2-y^2}$.

*17. $\dfrac{x-y}{x+y} + \dfrac{y^2+3xy}{y^2-x^2} + \dfrac{x+y}{x-y}$.

*18. $\dfrac{x+2y}{x+y} + \dfrac{2(y^2-4xy)}{y^2-x^2} - \dfrac{3y}{x-y}$.

19. $\dfrac{x^2-y^2}{xy} - \dfrac{xy-y^2}{xy-x^2}$.

*20. $\dfrac{x^2+y^2}{x^2-y^2} + \dfrac{x}{x+y} + \dfrac{y}{y-x}$.

Solve the following equations:

21. $\dfrac{x^2-5x}{x^2-4x-5} = \dfrac{2}{3}$

22. $\dfrac{3x^2+6x}{x^2+4x+4} = \dfrac{7}{3}$.

23. $\dfrac{x^2+5x+6}{x^2-1} \cdot \dfrac{x^2-2x-3}{x^2-9} = 5$.

24. $\dfrac{2x^2+13x+15}{4x^2-9} \div \dfrac{2x^2+11x+5}{4x^2-1} = 5$.

25. $\dfrac{2x^2-x-1}{2x^2+5x+2} \times \dfrac{4x^2+x-14}{16x^2-49} = \dfrac{1}{5}$.

* First combine the fractions whose denominators are of the first degree.

26. $\dfrac{1}{x+1}+\dfrac{1}{(x+1)(x+2)(x+3)}=\dfrac{1}{(x+1)(x+2)}-\dfrac{1}{3-x}.$

27. $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}=\dfrac{1}{11}.$

28. $\dfrac{x}{(x-3)(x+1)}-\dfrac{3}{(x+2)(x-3)}+\dfrac{1}{(x+2)(x+1)}=\dfrac{2}{3}.$

29. $\dfrac{1}{(3-x)^3(x-2)}+\dfrac{1}{(2-x)^3(3-x)^2}+\dfrac{1}{(5x-6-x^2)(x-2)}=0.$

*30. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}-\dfrac{4-20x}{4x^2-1}=\dfrac{x-2}{x+2}+\dfrac{x+2}{2-x}-\dfrac{2}{4-x^2}.$

31. $\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}=\dfrac{2}{11(x-4)}.$

32. $\dfrac{1}{x^2-7x+12}+\dfrac{1}{5x-x^2-6}=\dfrac{4}{5(3-x)(4-x)}.$

33. $\dfrac{1}{2x^2-x-1}+\dfrac{1}{3-x-2x^2}+\dfrac{1}{2(2x+1)(1-x)}=0.$

34. $\dfrac{1}{1+x-2x^2}+\dfrac{3}{6x^2-x-2}=\dfrac{1}{(2x+1)(1-x)}.$

35. $\dfrac{2}{x}+\dfrac{3}{1-2x}+\dfrac{3-2x}{4x^2-1}=\dfrac{1}{x-2x^2}.$

36. $\dfrac{1}{x-1}-\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}=\dfrac{x-1}{(x-x^2)(x^2+x-2)}.$

37. $\dfrac{5}{2(x+1)}-\dfrac{1}{10(x-1)}-\dfrac{24}{5(2x+3)}=\dfrac{5}{2x^2+5x+3}.$

38. $\dfrac{4}{4-7a-2a^2}+\dfrac{3}{10a^2+a-3}+\dfrac{a}{(4+a)(3+5a)}=0.$

39. $\dfrac{5}{5+x-18x^2}-\dfrac{2}{2x^2+5x+2}=\dfrac{3x}{(2+x)(5-9x)}.$

40. $\dfrac{1}{x+1}-\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+1)(x+2)(x+3)}=\dfrac{12}{(x+1)(x+3)}.$

* Simplify each member separately before clearing of fractions.

Modified Methods of Reduction.

241. There are certain types of fractional expressions and equations, of frequent occurrence, which can be much more easily reduced by modifying somewhat the ordinary straightforward method. For instance, examples 12, 13, 14, 16, and 26 in the preceding exercise.

Model F.—Reduce

$$\frac{1}{3-x}+\frac{1}{3+x}+\frac{6}{9+x^2}+\frac{54}{81+x^4}.$$

In this example, if we combine the first two fractions, then that result with the third, and finally that result with the fourth, the work can all be done mentally.

Sometimes the denominators are seen to belong together naturally in pairs:

Model G.

$$\frac{3}{2x-3}-\frac{2}{x+1}-\frac{3}{2x+3}+\frac{2}{x-1}$$

$$\equiv\frac{18}{(2x-3)(2x+3)}+\frac{4}{(x+1)(x-1)}$$

$$\equiv\frac{18x^2-18+16x^2-36}{(4x^2-9)(x^2-1)}\equiv\frac{34x^2-54}{4x^4-13x^2+9}.$$

Model H.—Reduce

$$\frac{1}{x-2}+\frac{1}{x-3}=\frac{1}{x-6}+\frac{1}{x-7}.$$

$$\text{②}\quad \frac{1}{x-2}-\frac{1}{x-7}=\frac{1}{x-6}-\frac{1}{x-3} \qquad \text{①}\quad -\frac{1}{x-3}-\frac{1}{x-7}$$

$$\text{③}\quad \frac{-5}{x^2-9x+14}=\frac{3}{x^2-9x+18} \qquad \text{same as ②}$$

④ $-5(x^2-9x+18)=3(x^2-9x+14)$

③ $\times (x^2-9x+14)(x^2-9x+18)$

⑤ $-5x^2+45x-90=3x^2-27x+42$ same as ④

⑥ $8x^2-72x+132=0$ ⑤ $+5x^2-45x+90$

⑦ $2x^2-18x+\ \ 33=0$ ⑥ $\div 4$

Whence by the Quadratic Formula $x=3.218$; $x=1.718$.

The advantage of rearranging the equation as in ② is that the x-terms destroy each other, and we save such multiplications as

$$(2x-5)(x^2-13x+42) \text{ and } (2x-13)(x^2-5x+6).$$

Model I.—Reduce

$$\frac{x-2}{x-3}+\frac{x-1}{x-2}=\frac{x-6}{x-7}+\frac{x-5}{x-6}.$$

If we divide the numerator of the first fraction by its denominator, the quotient is 1 and the remainder 1; so that the fraction reduces to the **mixed number** $1+\dfrac{1}{x-3}$.

② $1+\dfrac{1}{x-3}+1+\dfrac{1}{x-2}=1+\dfrac{1}{x-7}+1+\dfrac{1}{x-6}$ same as ①

③ $\dfrac{1}{x-3}+\dfrac{1}{x-2}=\dfrac{1}{x-7}+\dfrac{1}{x-6}$ ② -2

The rest of the reduction proceeds as in the preceding example.

Model J.—Reduce

$$\frac{x^2+7x+15}{2x^3+13x^2+24x-10}=\frac{2x^2+3x+3}{4x^3+4x^2+5x-4}$$

The reciprocals of these two equal fractions are equal,—

$$ⓐ\ \frac{2x^3+13x^2+24x-10}{x^2+7x+15}=\frac{4x^3+4x^2+5x-4}{2x^2+3x+3} \qquad 1 \div ①$$

Reducing to mixed numbers, by carrying out the division indicated by each fraction,—

$$③\ 2x-1+\frac{x+5}{x^2+7x+15}=2x-1+\frac{2x-1}{2x^2+3x+3}$$

same as ②

$$④\ \frac{x+5}{x^2+7x+15}=\frac{2x-1}{2x^2+3x+3} \qquad ③-2x-1$$

$$⑤\ \frac{x^2+7x+15}{x+5}=\frac{2x^2+3x+3}{2x-1} \qquad 1\div ④$$

$$⑥\ x+2+\frac{5}{x+5}=x+2+\frac{5}{2x-1} \qquad \text{same as } ⑤$$

$$⑦\ \frac{5}{x+5}=\frac{5}{2x-1} \qquad ⑥-x-2$$

Whence $x+5=2x-1$ and $x=6$.

Care must be taken not to apply the method of the example just preceding to equations where there are more than one term in either member; that is, if

$$\frac{3}{x-2}+\frac{x-3}{x^2-5x+1}=\frac{x^2+3}{x^3+5x+2}$$

IT IS NOT TRUE THAT

$$\frac{x-2}{3}+\frac{x^2-5x+1}{x-3}=\frac{x^3+5x+2}{x^2+3}.$$

In examples like number 1 of the next exercise clear of the numerical denominators first.

EXERCISE CXIII.

Reduce to the simplest form:

1. $\dfrac{11x-7}{12}+\dfrac{5x-7}{3}=\dfrac{7x-3}{x-1}+\dfrac{3x-5}{5}.$

2. $\dfrac{1}{2x-3}+\dfrac{1}{2x+3}-\dfrac{4x}{4x^2+9}+\dfrac{17}{16x^4+81}.$

3. $\dfrac{3x}{6x-4}-\dfrac{x}{3x+2}-\dfrac{3x^2+2}{18x^2+8}+\dfrac{4(3x+2)}{81x^4+16}.$

4. $\dfrac{3}{2x-3}+\dfrac{7}{3+x}+\dfrac{2}{2x+3}-\dfrac{4}{3-x}.$

5. $\dfrac{3}{x+1}-\dfrac{1}{2(x+2)}-\dfrac{x+4}{(x+1)(x+2)}+\dfrac{x+1}{(x+2)(x+3)}.$

6. $\dfrac{1}{x-3}-\dfrac{4}{x-1}+\dfrac{5}{x}-\dfrac{4}{x+1}+\dfrac{1}{x+3}.$

7. $\dfrac{2x+1}{5}+\dfrac{4x-7}{x-4}+\dfrac{x-6}{3}=\dfrac{x+5}{2x-8}+\dfrac{3x-2}{3}.$

8. $\dfrac{x-1}{x-3}+\dfrac{20x+11}{3}=\dfrac{x-11}{3x-9}+\dfrac{41x-4}{6}.$

9. $\dfrac{1}{4-x}+\dfrac{2}{2x-3}=\dfrac{2}{7-2x}+\dfrac{1}{x-1}.$

10. $\dfrac{1}{x}+\dfrac{1}{5-x}-\dfrac{1}{x+1}=\dfrac{1}{4-x}.$

11. $\dfrac{2}{x-2}-\dfrac{2}{x-7}-\dfrac{2}{x-1}=\dfrac{2}{6-x}.$

12. $\dfrac{1}{x+9}+\dfrac{1}{x+5}=\dfrac{1}{x+8}+\dfrac{1}{x+6}.$

13. $\dfrac{3x^2-x-25}{9x^3-6x^2-71x+18}=\dfrac{x^2+7x-1}{3x^3+20x^2-10x+6}.$

14. $\dfrac{4x^2-4x-3}{4x^3+8x^2-10x-3}=\dfrac{2x^2+3x+4}{2x^3+9x^2+13x+15}.$

15. $\dfrac{7x^2+10x-1}{77x^3+103x^2-14x-3}=\dfrac{3x^2+22x+32}{33x^3+239x^2+410x-23}.$

16. $\dfrac{6x^2-x+13}{6x^3+11x^2+14x+24}=\dfrac{4x^2+12x+20}{4x^3+20x^2+46x+45}.$

17. $\dfrac{3x+13}{x+3}+\dfrac{x-1}{x+1}=\dfrac{4x+17}{2x+6}+\dfrac{8x+3}{4x+4}.$

18. $\dfrac{3x-8}{x-3}+\dfrac{4x-17}{x-4}=\dfrac{2x-13}{x-7}+\dfrac{5x-41}{x-8}.$

19. $\dfrac{8x-27}{2x-7}+\dfrac{2x-9}{x-5}-\dfrac{6x-8}{2x-3}=\dfrac{3x-17}{x-6}.$

20. $\dfrac{15x-2}{3x-1}+\dfrac{12x-41}{3x-11}=\dfrac{14x-19}{2x-3}+\dfrac{4x-8}{2x-5}.$

21. $\dfrac{1}{x-a}-\dfrac{1}{x+a}-\dfrac{2a}{x^2+a^2}-\dfrac{4a^3}{x^4+a^4}.$

22. $\dfrac{1}{x+4}+\dfrac{1}{x-16}=\dfrac{1}{x-5}+\dfrac{1}{x-7}.$

23. $\dfrac{4x-3}{7}-\dfrac{3x+4}{2x-1}+\dfrac{x-5}{7}=\dfrac{3x+4}{2}-\dfrac{10x-4}{7}+1\tfrac{6}{7}.$

24. $\dfrac{66x-49}{6x-5}+\dfrac{4x-5}{x-1}=\dfrac{14x-19}{2x-3}+\dfrac{24x-43}{3x-5}.$

25. $\dfrac{x^2+28x-55}{7x^3+201x^2-244x-245}=\dfrac{3x^2+20x-47}{21x^3+155x^2-226x-209}.$

26. $\dfrac{x^2+2x-3}{x^3+3x^2-3x-3}=\dfrac{67x+133}{67x^2+200x-1}.$

27. $\dfrac{x^2+x-6+1}{x-2}+\dfrac{8x^2-18x-9}{4x+1}$

$$=\dfrac{24x^2-55x+26}{8x-13}-\dfrac{1}{2x-2}.$$

28. $\dfrac{2x^4+21x^2-2x+120}{x^4+11x^2+64}=\dfrac{4x^3+6x^2-2x+105}{2x^3+3x^2+56}.$

The Principal Dividing Line.

242. A fraction in which the numerator or the denominator or both contain a fractional term is called **a complex fraction.**

For example, the ratio $\frac{2}{3} : 5$ may be written as a complex fraction; so also $2 : \frac{3}{5}$. When so written these expressions cannot be distinguished except by the **Principal Dividing Line** of the fraction.

$$\text{Thus } \frac{\frac{2}{3}}{5} = \frac{2}{15}; \text{ while } \frac{2}{\frac{3}{5}} = \frac{10}{3}.$$

243. In a complex fraction **everything above the principal dividing line constitutes the dividend, everything below constitutes the divisor.** Plus and minus and equal signs which precede or follow a complex fraction must be on the same level as the principal dividing line of the fraction.

244. In most cases the readiest way to simplify a complex fraction is to take advantage of the principle that multiplying the numerator and the denominator by the same number does not change the value of a fraction; and to choose such a multiplier as will render both numerator and denominator integral.

Model K.

$$\frac{x - \frac{y^3}{x^2}}{x^2 - y^2} = \frac{x^3 - y^3}{x^2(x^2 - y^2)} = \frac{x^2 + xy + y^2}{x^2(x + y)}.$$

245. Sometimes one prefers to simplify the numerator and the denominator separately, reducing them each to a single fractional term, and then dividing the numerator by the denominator.

Model L.

$$\frac{\frac{a}{p}-\frac{b}{q}}{\frac{p}{a}-\frac{q}{b}} \equiv \frac{\frac{aq-bp}{pq}}{\frac{pb-aq}{ab}} \equiv \frac{aq-bp}{pq} \div \frac{bp-aq}{ab} \equiv -\frac{ab}{pq}.$$

EXERCISE CXIV.

Simplify:

1. $\dfrac{\frac{x}{y}-\frac{y}{x}}{x^3-y^3}.$

2. $\dfrac{\frac{a}{b}-\frac{c}{d}}{\frac{x}{y}-\frac{p}{q}}.$

3. $\dfrac{\frac{a}{x}-\frac{b}{y}}{\frac{x}{y}-\frac{a}{b}}.$

4. $\dfrac{\frac{x}{3}-\frac{y}{5}}{\frac{3}{x}-\frac{y}{5}}.$

5. $\dfrac{\frac{x^3}{27}+\frac{a^3}{8}}{9a^2-4b^2}.$

6. $\dfrac{\frac{1}{b^2}-\frac{1}{a^2}-\frac{5}{6ab}}{\frac{1}{b^2}+\frac{1}{a^2}-\frac{13}{6ab}}.$

7. $\dfrac{\frac{x-2}{x-3}}{x^2-9}.$

8. $\dfrac{\frac{x-2}{x^2-5x+6}}{x^2-9}.$

9. $\dfrac{1+\frac{1}{x-5}}{1+\frac{20-9x}{x^2}}.$

10. $\dfrac{\frac{ax-x^2}{(a+x)^2}}{\frac{x^2}{a^2-x^2}}.$

11. $\dfrac{\frac{x^3-8}{x^2-4}}{x-2} \times \dfrac{\frac{x^3-8}{x^2-4}}{x-2}.$

12. $\dfrac{\frac{\frac{1}{x}-\frac{\frac{1}{x}}{y}}{y}}{y^3-\frac{1}{y}}.$

13. $\dfrac{1-\frac{a+b}{a-b}}{1+\frac{a-b}{a+b}}.$

14. $\dfrac{a^2-b^2-c^2-2bc}{\frac{a+b+c}{a+b-c}}.$

15. $\dfrac{\frac{x^2+y^2-z^2-2xy}{2z}}{1+\frac{2z}{x-y-z}}$

16. $\dfrac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}.$

17. $$\frac{x+\frac{x}{5}+\frac{1}{3}}{\frac{1}{5}+\frac{5}{3x}+\frac{3}{x}}.$$

18. $$\frac{\frac{a+b}{a-b}+\frac{x+y}{x-y}}{\frac{a+b}{x+y}-\frac{a-b}{x-y}}.$$

19. $$\frac{\frac{x+3}{x-3}+\frac{2x+1}{2x-1}}{\frac{x+3}{2x+1}-\frac{x-3}{2x-1}}.$$

20. $$\frac{\frac{a+b}{x+y}+\frac{a-b}{x-y}}{\frac{a+b}{x-y}-\frac{a-b}{x+y}}.$$

Continued Fractions.

246. Complex fractions of the following peculiar type are called continued fractions :

Model M.

$$\cfrac{2}{3+\cfrac{1}{2+\cfrac{3}{4-\cfrac{1}{4-\frac{1}{5}}}}}$$

To simplify this expression, rewrite it as far as the last dividing line which indicates a complex fraction; substitute for that its simplest form, and repeat the same process as often as necessary.

$$\cfrac{2}{3+\cfrac{1}{2+\cfrac{3}{4-\cfrac{1}{4-\frac{1}{5}}}}}=\cfrac{2}{3+\cfrac{1}{2+\cfrac{3}{4-\frac{5}{19}}}}=\cfrac{2}{3+\cfrac{1}{2+\frac{57}{71}}}$$

$$=\frac{2}{3+\frac{71}{199}}=\frac{398}{598}=\frac{199}{299}$$

EXERCISE CXV.

Simplify :

1. $\dfrac{1}{1-\dfrac{1}{1-x}}$ 2. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$ 3. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{1}{x+\dfrac{1}{x}}}}$

4. $\dfrac{a}{1+\dfrac{1}{a-\dfrac{1}{1-\dfrac{1}{a}}}}$ 5. $\dfrac{1+x}{x+\dfrac{1}{1+\dfrac{x+1}{2-x}}}$

6. $\dfrac{\dfrac{2a}{25}-\dfrac{25}{2a}}{15-\dfrac{5}{5+\dfrac{2a-3}{1-\dfrac{2a}{5}}}}$ 7. $\dfrac{\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}{\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}}$

8. In the expression $\dfrac{1}{2+\dfrac{x}{3}}$ substitute for x the expression $\dfrac{1}{3+\dfrac{1}{4+\frac{1}{5}}}$, and reduce to simplest form.

9. Find the value of the expression $\dfrac{x}{y}:\dfrac{x'}{y'}$ when

$$x=1+\frac{1}{2+\frac{1}{3+\frac{1}{4}}} \text{ and } y=1-\frac{1}{2+\frac{1}{3-\frac{1}{4}}};$$

x' is obtained from x by omitting the fraction indicated by

the lowest dividing line, and y' is obtained from y in the same way.

10. Find the value of the ratio $\frac{2x}{y}:\frac{x}{2y'}$, when

$$x = a + \cfrac{1}{b + \cfrac{1}{c}};$$

y is the result obtained by substituting $c + \frac{1}{a}$ for c in the expression for x; x' is the result obtained by interchanging b and c in the expression for x, and y' is the result of the same change in y.

Another Application of L. C. M.

247. If we have an equation in the form of an integral expression equal to zero, the answers can be found if the equation can be factored. The answers of two or more such equations may all be found in the equation formed by setting the L. C. M. of the expressions equal to zero.

Model N.—The equations following have the answers set down beside them:

$$\begin{array}{rl} x^2-3x=0 & x=0;\quad x=3 \\ x^2-5x+6=0 & x=2;\quad x=3 \\ x^2-9=0 & x=3;\quad x=-3 \end{array}$$

The L. C. M. of the expressions x^2-3x, x^2-5x+6, and x^2-9, is $x(x-3)(x+3)(x-2)$; and the equation

$$x(x-3)(x+3)(x-2)=0$$

has answers 0, 2, 3, -3.

EXERCISE CXVI.

For each of the following sets of equations construct a new equation, among whose answers can be found all the answers of each given equation; and let the resulting equation be the one of lowest degree which can satisfy this condition:

1. $2x^2 = 3x + 2; \quad 4x^2 = 1; \quad x^2 + 2 = 3x.$
2. $8x^3 - 27 = 0; \quad 4x = \dfrac{9}{x}$
3. $4x^2 = \pm 1; \quad 2x = 1.$
4. $x^2 = x + 20; \quad x^2 = 12 - x; \quad x^2 = 8x - 15.$
5. $x^2 = \frac{9}{4}(x+1); \quad x^2 = 3(x+1) + \dfrac{x}{4}; \quad x^2 = 7x - 12.$
6. $x^3 = 8a^3; \quad x + 2a + \dfrac{4a^2}{x} = 0; \quad x = \dfrac{4a^2}{x}.$
7. $(3x-2) : (x-1) = 5 : 2x; \quad 6x^2 + 5 = 13x; \quad 4x^2 = 1.$
8. $6x^2 = x + 2; \quad 18x(1-x) = 3x^2 + x + 2; \quad 5x(3x+1) = x + 1.$
9. $x^3 = 27; \quad x^2 = 15x - 36; \quad x^3 - 3x^2 - 2x + 6 = 0.$
10. $5x^2 + 19x = 4; \; 10x^2 + 19x = 3 + 6x;$
$$15x^2 + 19x = 13x^2 + 4(2x-3).$$

Tests for Simple Factors.

248. Sometimes it is not possible to factor all the expressions by inspection; and in such cases it often serves to get the factors of one expression and to try them as divisors of the other expressions.

Model 0.—In the expressions

$$2x^2 + 7x + 5; \quad 2x^3 + x + 3; \quad x^3 - 6x - 5$$

the factors of the first expression are $(2x+5)(x+1)$; of

these $2x + 5$ is evidently not contained in either of the other expressions; by substituting $x = -1$ in the other expressions, we reduce each to zero, and hence conclude that $x + 1$ is a factor of each. Dividing we find the factors as follows:

$$2x^3 + x + 3 \equiv (x + 1)(2x^2 - 2x + 3);$$
$$x^3 - 6x - 5 \equiv (x + 1)(x^2 - x - 5).$$

Hence the L. C. M. would be

$$(x + 1)(2x + 5)(2x^2 - 2x + 3)(x^2 - x - 5).$$

EXERCISE CXVII.

Rearrange the equations in each example so that each shall be an integral algebraic expression equated to zero. Then find the L. C. M. of their first members:

1. $x^2 - 4 = 0$; $x^3 + x = 10$.
2. $x^2 - 7x - 30 = 0$; $2x^3 + 15 = 13x$; $3x^3 = 29x + 6$.
3. $2x^2 = 9x + 5$; $x(x^2 - 9) = 20(x - 1)$; $x^2(2x - 1) = 5(2x^2 - 5)$.
4. $x^2 = \frac{1}{2}(x + 10)$; $x(x^2 - 2) = \frac{2}{3}(x - 4)$; $3x^3 + 4 = 7x^2$.
5. $x - 2 = \frac{3}{8}x^2$; $x^4 - 12 = 15x^2 + x$; $x^4 = 16 + 2x^2 + 3x^3$.
6. $x^2 = \frac{2}{5}(9x + 4)$; $x^3 - 6x^2 + 32 = 0$; $15x^3 - 19x^2 + 4 = 0$.
7. $x^3 = 8$; $x^4 + 4x^2 + 16 = 0$; $x^3 = x^2 + 4$.
8. $1000y^3 = 1$; $10y^3 = 99y^2 - 1$.
9. $2x^3 - 6x^2 + 3x = 9$; $2x^3 = 19x - 3$.
10. $2x^3 - 5x^2 = 16x - 40$; $x^4 - 6x^2 = x^3 - 8x + 16$; $2x^3 = x^2 + 25$.

Relation between L. C. M. and H. C. F.

249. The L. C. M. of any two expressions may be obtained by dividing one of them by their H. C. F. and multiplying the quotient by the other.

For since the factors of the H. C. F. are contained in each expression, the first expression contains all the factors

of the second except those that are not included in the H. C. F.

Model P.—Of the two expressions $2x^5 - 11x^2 - 9$ and $4x^5 + 11x^4 + 81$ the H. C. F. is $x^2 + 2x + 3$; the quotient obtained by dividing the first expression, $2x^5 - 11x^2 - 9$, by the H. C. F., $x^2 + 2x + 3$, is $2x^3 - 4x^2 + 2x - 3$; so the L. C. M. is $(2x^3 - 4x^2 + 2x - 3)(4x^5 + 11x^4 + 81)$; if we divided the second expression by the H. C. F. we should obtain for the L. C. M. $(4x^3+3x^2-18x+27)(2x^5-11x^2-9)$; in either case the product would be

$$8x^8+6x^7-36x^6+10x^5-33x^4+162x^3-324x^2+162x-243.$$

EXERCISE CXVIII.

Find the L. C. M. of the following expressions:

1. $x^{10} + x^5y^5 + y^{10}$; $x^{12} + x^9y^3 + x^6y^6 + x^3y^9 + y^{12}$.
2. $x^4 + x + 6$; $x^3 + 2x^2 + 9$.
3. $2x^4 + 66xy^3 - 20y^4$; $9x^3 + 30xy^2 - 3y^3$.
4. $x^5+15x^2y^6-11xy^8+15y^{10}$; $5x^5+11x^4y^2+20x^2y^6+9y^{10}$.
5. $42a^4x + 13a^3x^4 - x^{13}$; $36a^5 - x^{15} + 8a^2x^9 - a^3x^6$.

L. C. M. of Three Expressions.

250. In finding the L. C. M. of three or more expressions, one finds first the L. C. M. of the first two; then it is necessary to multiply by all the factors of the third expression that are not contained in the first two.

If the three expressions are represented by X, Y, and Z, and the L. C. M. of X and Y by m, we must multiply m by all the factors of Z that are not to be found in m already. To find these, we divide Z by the H. C. F. of m and Z.

Model Q.—In the three expressions

$$x^3 - y^3;\ x^2 - y^2;\ x^4 - y^4$$

the L. C. M. of the first two is $(x^3-y^3)(x+y)$; the H. C. F. of this and the third expression is x^2-y^2; hence the L. C. M. of all is $(x^3-y^3)(x+y)(x^2+y^2)$.

EXERCISE CXIX.

Find the L. C. M. of the five sets of expressions in Exercise CVI; also of the following five:

1. $4x^3-7x-3$; $2x^3-7x^2+9$; $4x^3-16x^2+9x+9$.
2. $x^3-6x^2+11x-6$; x^3-2x^2-5x+6; $3x^4-13x^3+52x-48$.
3. $x^3+2x-12$; x^3-x^2-18; x^3-8x-3.
4. $4x^3-7x-3$; $1-6x^3-7x^2$; $5x-4x^3-6$.
5. $x^4-3x-20$; x^4-4x^2+15x; $25-4x^2+3x^3$.

Rearrange the equations in each of the following examples so that each shall consist of an integral algebraic expression equated to zero; then find the L. C. M. of those expressions:

6. $\dfrac{x^3+11x}{6}=x^2+1$; $x^3-x-24=9x^2-27x$; $\dfrac{19x-12}{x^2}=8-x$

7. $x^2-6x=\dfrac{24-2x-x^2}{x-4}$; $\dfrac{x}{x^2+3}=\dfrac{10}{x^2+31}$; $11x^2-x^3=38x-40$

8. $3x^3-13x^2+23x-21=0$
$6x^3+x^2=44x-21$
$46x+7=9x^3$

9. $\dfrac{x^4-1}{x}=(3x+1)(x-1)$
$x^2=\dfrac{5-2x}{3x}$

10. $x^4+8x^2+28x=7x^3+48$; $x^3-14=8x^2-19x$;
$5x^3+28=17x^2$

CHAPTER X.

INDICES; SURDS; ROOTS.

251. A **Power** is a number whose factors are equal.

One of the equal factors is called a **Root.**

The number of equal factors is called the **Index.**

$$a^3a^5 \equiv aaa.aaaaa \equiv a^8$$

Law I. $$a^xa^y \equiv a^{x+y}$$

$$a^5 \div a^3 \equiv \frac{aaaaa}{aaa} \equiv aa \equiv a^2$$

Law II. $$a^x \div a^y \equiv \frac{a^x}{a^y} \equiv a^{x-y}.$$

When $y = x$ the quotient in (II) becomes a^0;

When $y > x$, the quotient in (II) being a^{x-y}, the index $x - y$ is a minus number.

Thus $$a^3 \div a^3 = a^{3-3} = a^0$$
$$a^3 \div a^8 = a^{3-8} = a^{-5}$$

252. For these new values of the index the old definitions will not apply; but if we consider them subject to laws (I) and (II) we can interpret them in terms of the more familiar symbols.

$$a^0a^m \equiv a^{0+m} \equiv a^m$$

Law III. $a^0 \equiv \dfrac{a^m}{a^m} \equiv 1$

$$a^{-3}a^3 \equiv a^{-3+3} \equiv a^0 \equiv 1$$

$$a^{-3}a^3 \equiv 1;\ a^{-3} \equiv \frac{1}{a^3};\ \ a^3 \equiv \frac{1}{a^{-3}}$$

$$a^{-m}a^m \equiv a^{-m+m} \equiv a^0$$

Law IV. $a^{-m}a^m \equiv 1;\ a^{-m} \equiv \dfrac{1}{a^m};\ a^m \equiv \dfrac{1}{a^{-m}}$

EXERCISE CXX.

State in the simplest form the numerical value of the following expressions:

1. 3^{-3}. **2.** $\left(\dfrac{1}{2}\right)^{-2}$. **3.** $(.01)^{-1}$. **4.** $\dfrac{1}{2^{-3}}$. **5.** $\dfrac{1}{(\sqrt{2})^{-4}}$.

6. $256(4)^{-3}$. **7.** $(16)^2(6)^{-3}$. **8.** $\left(\dfrac{1}{10}\right)^{-3}\left(\dfrac{1}{5}\right)$. **9.** $9\dfrac{6^{-2}}{10^{-3}}$.

10. $\left(\dfrac{1}{.25}\right)^{-1}$.

What is the reciprocal of each of the following expressions?

11. 2. **12.** 3. **13.** 10. **14.** 1. **15.** 100. **16.** $\frac{1}{3}$.

17. $\frac{2}{3}$. **18.** $\frac{5}{6}$. **19.** .3. **20.** .001. **21.** 2.5. **22.** $3\frac{1}{7}$.

23. 100.1. **24.** 1.1. **25.** 33.33. **26.** a. **27.** $\dfrac{a}{x}$.

28. $\dfrac{a^2}{3}$. **29.** $\dfrac{a-b}{c}$. **30.** $x + y^2$.

253. From (IV) it follows that the reciprocal of any power is the same power with the sign of its index changed.

EXERCISE CXXI.

State the reciprocals of the following expressions without using the fractional form:

1. x^3; x^{-y}; $\dfrac{1}{x^2}$; $\dfrac{1}{x^{-3}}$; x^{1-y}. 2. $\dfrac{x}{7}$. 3. $(a^2+x^2)^2$.

4. $(a^{-3}-x^{-1})^{-3}$. 5. $(a^2+x^2-y^2)^{-3}$. 6. $(x^2+y^2)^{x^2-y^2}$.

7. 16^{-2}. 8. 2^{a-b}. 9. $a-b$. 10. $x^{-2}+y^3$.

Reversal of Signs.

254. The law of reversal of signs in subtraction has a close analogy in multiplication:

In subtraction: **To use an expression as a TERM of the SUBTRAHEND is the same as to use its NEGATIVE as a TERM of the MINUEND.**

In division: **To use an expression as a FACTOR of the DIVISOR is the same as to use its RECIPROCAL as a FACTOR of the DIVIDEND.**

Or, conversely:

Any TERMS of an expression may be replaced by a SUBTRAHEND containing the NEGATIVES of those TERMS.

Any FACTORS of an expression may be replaced by a DIVISOR containing the RECIPROCALS of the FACTORS.

255. Dividend, numerator, and antecedent are in Algebra interchangeable names; so also are divisor, denominator, and consequent; every example of division being a fraction or a ratio, according to circumstances which are not always apparent.

EXERCISE CXXII.

Transfer all x's and y's to the divisor, and all a's, b's, c's, and d's to the dividend, wherever they occur in the following fractions (or ratios), changing the index when necessary:

1. $\frac{x^{-2}}{a^{-2}}$. 2. $\frac{x^2}{a^2}$. 3. $\frac{x^2y}{a^{-2}b}$. 4. $\frac{axy}{b^2c^{-3}}$. 5. $a^{-2}x^2 : b^2cy^{-2}$.

6. x. 7. $\frac{x^2}{(a+b)^{-1}}$. 8. $\frac{y+x}{a^{-3}}$. 9. $\frac{x^{-2}}{1+a^2}$.

10. $\frac{x+y^{-2}}{a^{-1}+b^2}$. 11. $\frac{ax(x+y)}{by^2(c^2-a^3)}$.

256. Model A. $\frac{3a^2b^{-3}c^0d^{-1}}{2^{-3}g^{-1}x^3yz^{-2}} \equiv \frac{3.2^3a^2gz^2}{b^3dx^3y} \equiv \frac{24a^2gz^2}{b^3dx^3y}$.

EXERCISE CXXIII.

Change to expressions that have no zero or negative indices:

1. a^2x^{-3} 2. $\frac{a^2}{x^{-3}}$. 3. $\frac{a^2b^2c^{-3}}{x^3y^{-2}z^{-2}}$. 4. $a^3b^2cd^0e^{-1}f^{-2}$.

5. $\frac{x^3y^{-3}}{a^{-2}}$. 6. $\frac{5x^{-2}}{3y^2z^{-3}}$. 7. $\frac{ab^{-3}}{3y^{-2}}$. 8. $\frac{(2x^2)^{-3}}{8x^{-4}y^{-2}}$.

9. $\frac{2^{-3}x^{-2}y^2}{16a^{-3}b^2}$. 10. $\frac{(6ay)^{-5}}{(3a^2)^{-2}(2y^3)^{-4}}$.

Simplify the following expressions:

11. $x^{-2}+y^2$. 12. $a^{-1}-b^{-1}$. 13. $\frac{x^2-y^2}{x^{-2}-y^{-2}}$.

14. $\frac{x^2-5x+6}{1-9x^{-2}}$. 15. $\frac{a^{-3}+b^{-3}}{a-b+b^2a^{-1}}$.

16. $\dfrac{ab^{-1} + b(a + b)^{-1}}{a(a - b) \quad - a^{-1}b}$.

17. $(a^{-2} - b^{-2})\{a(a + b)^{-1} + b(a - b)^{-1}\}$.

18. $(x^{-6} - y^{-6})(x^{-3} - y^{-3})^2$.

19. $(x - y^3x^{-2})(x^3 + y^3)^{-1}(xy^{-1} - x^0y^0 + x^{-1}y)$
$(x^{-1}y^{-3} + x^{-2}y^{-2} + x^{-3}y^{-1})^{-1}$.

20. $(xy^{-1} - yx^{-1})(y^{-1} - x^{-1})^{-2}$.

When a Root is itself a Power.

257. In $4 \times 4 \times 4 = 64$; $2^2 \times 2^2 \times 2^2 = 64$ there are two indices to consider . . . and factors within factors.

$$(a^3)^5 = a^3a^3a^3a^3a^3 = aaaaaaaaaaaaaaa = a^{15}.$$

Here for each of the large equal factors there are three of the smaller ones; in all five times three smaller factors.

$$(a^x)^6 = a^xa^xa^xa^xa^xa^x = a^{6x}.$$

Here for each of the large equal factors there are x of the smaller ones; in all $6x$ of the smaller factors.

Law V. $\qquad (a^p)^q \equiv a^{pq}$.

258. Referring to (v), we have a **power** a^{pq}

of which a^p is a **root**

the **index** being q.

This relation may be written as in (v), but also

$$a^p = \sqrt[q]{a^{pq}}.$$

Similarly $a^2 = \sqrt[3]{a^6}$, since a^6 may be arranged in three equal factors thus : $aa.aa.aa$; and again $a^x = \sqrt[7]{a^{7x}}$, since a^{7x} may be arranged in seven equal factors, each of which

contains x factors a. In general, if we have $\sqrt[q]{a^p}$, and if p is a multiple of q, we may arrange the p factors of a^p in q equal groups; each group will contain $\frac{p}{q}$ factors a; and in that case, IF p IS A MULTIPLE OF q,

$$\sqrt[q]{a^p} \equiv a^{\frac{p}{q}}.$$

EXERCISE CXXIV.

Express as powers of prime numbers:

1. $(2^{13})(8^4)(4^{326})(16^{200})$.
2. $(9^{211})(27^{108})(6^{100})(8^{33})$.
3. $(12)^3(36)^7(192)^{43}$.
4. $(200)^4(135)^{38}(225)^{51}$.
5. $(45)^{11}(675)^{31}(375)^{28}$.
6. $(8)^{23} \div (16)^{11}$.
7. $\{(24)^{12}(54)^6\}^{11}$.
8. $[(1000)^8 \div (25)^{10}]^{18}$.
9. $\sqrt[5]{3^{32} \,.\, 9^{27} \,.\, 81}$.
10. $\sqrt[5]{(12)^{13}(18^6) \div 2^7}$.

Fractional Indices.

259. If p is not an exact multiple of q, then the expression $a^{\frac{p}{q}}$ becomes meaningless, as we cannot have a fractional number of factors. We are at liberty to assume that laws (I), (II), and (V) apply to these new indices, and under that assumption to seek an interpretation for them.

$$a^{\frac{2}{3}}a^{\frac{2}{3}}a^{\frac{2}{3}} \equiv a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} \equiv a^{\frac{6}{3}} \equiv a^2.$$

Here $a^{\frac{2}{3}}$ appears as one of the three equal factors of a^2; that is, $a^{\frac{2}{3}} \equiv \sqrt[3]{a^2}$.

$$(a^{\frac{5}{7}})^7 \equiv a^5; \;\; a^{\frac{5}{7}} \equiv \sqrt[7]{a^5}$$

$$\left(a^{\frac{9}{x}}\right)^x \equiv a^9, \;\; a^{\frac{9}{x}} \equiv \sqrt[x]{a^9}$$

$$\left(a^{\frac{p}{q}}\right)^q \equiv a^p; \;\; a^{\frac{p}{q}} \equiv \sqrt[q]{a^p}$$

In general any power whose index is a fraction may be taken as a ROOT, and as many of them multiplied together as the denominator indicates; then the product so obtained will be a power whose index is the numerator.

$$\left(a^{\frac{p}{q}}\right)^q \equiv a^p; \text{ or, otherwise,}$$

Law VI. $$a^{\frac{p}{q}} \equiv \sqrt[q]{a^p}$$

260. By (VI) any root may be exhibited as a power with a fractional index.

261. Again

$$a^{\frac{p}{q}} \equiv \left(a^{\frac{1}{q}}\right)^p \equiv \sqrt[q]{(a)^p} \text{ as well as } a^{\frac{p}{q}} \equiv (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}.$$

A fractional index, then, indicates both a root and a power; the denominator indicates a root, and the numerator indicates a power; and either the root or the power may be found first.

EXERCISE CXXV.

Express with radical signs :

1. $a^{\frac{1}{2}}$. **2.** $x^{\frac{2}{3}}$. **3.** $(a-b)^{\frac{5}{2}}$. **4.** $a^{\frac{1}{2}}b^{\frac{3}{2}}$.
5. $a^{\frac{5}{6}}b^{\frac{3}{6}}$. **6.** $x^{\frac{3}{4}}y^{\frac{2}{4}}$. **7.** $x^{\frac{1}{4}}y^{\frac{1}{2}}$ (compare Ex. 6).
8. $p^{\frac{1}{6}}q^{\frac{1}{2}}$ (compare Ex. 5). **9.** $a^{\frac{1}{3}}x^{\frac{1}{2}}$. **10.** $(ax^2y^3)^{\frac{2}{5}}$.

Express with fractional exponents :

11. $\sqrt{a^3}$. **12.** $\sqrt[3]{a^2b}$. **13.** $\sqrt[4]{x^2a^3}$. **14.** $\sqrt[6]{ax^2y^3}$.

15. $\sqrt[3]{(a-b)^2}$. **16.** $\sqrt[3]{a^2-b^2}$. **17.** $\sqrt[4]{x^2y^6}$. **18.** $\sqrt[10]{x^4y^5}$.

19. $\sqrt[12]{a^2x^3z^4y^6}$. **20.** $\dfrac{\sqrt[6]{a^4b^3}}{\sqrt[4]{xy^3}}$.

Give the numerical value of each :

21. $8^{\frac{2}{3}}$. 22. $16^{\frac{3}{4}}$. 23. $32^{\frac{3}{5}}$. 24. $(\frac{4}{9})^{\frac{3}{2}}$. 25. $(\frac{27}{64})^{\frac{4}{3}}$.
26. $(.36)^{\frac{3}{2}}$. 27. $(.008)^{\frac{4}{3}}$. 28. $(1.728)^{\frac{2}{3}}$. 29. $-(15625)^{\frac{5}{6}}$.
30. $(256)^{-\frac{3}{8}}$. 31. $(\frac{32}{243})^{-\frac{2}{5}}$. 32. $(-8)^{-\frac{7}{3}}$. 33. $(64)^{-\frac{1}{6}}(\frac{1}{4})^{-2}$.
34. $(.0001)^{-\frac{1}{4}}$. 35. $(81)^{-\frac{3}{4}}(.3)^2$. 36. $(\frac{1}{3})^{-3}(243)^{-\frac{4}{5}}$.
37. $(\frac{1}{49})^{-\frac{1}{2}}(\sqrt{11})^{-2}$. 38. $(216)^{\frac{2}{3}}(\frac{1}{11})^{-2}(.1)^{-1}$.
39. $(.0625)^{-\frac{3}{4}}(\frac{1}{2})^{-3}$. 40. $\{(128)^{\frac{5}{7}}(\frac{1}{36})^{-\frac{3}{2}}32^{-\frac{2}{5}}6^{-2}\}^{-1}$.
41. $4^{.5}$. 42. $9^{2.5}$. 43. $16^{.75}$. 44. $32^{.8}$. 45. $256^{.125}$.
46. $(\frac{4}{9})^{1.5}$. 4 $(\frac{81}{16})^{1.25}$. 48. $(\frac{243}{32})^{-.6}$. 49. $(2.25)^{2.5}$.
50. $\left(\dfrac{a^{12}}{b^{16}}\right)^{3.75}$

Simplify :

51. $(ab^{\frac{1}{2}})^{\frac{1}{3}}$. 52. $(a^2b^{-1})^{\frac{1}{2}}$. 53. $(a^2b^{-3})^{\frac{1}{8}}$.
54. $(a^{\frac{1}{2}}b^{\frac{1}{4}})^{\frac{2}{3}}$. 55. $(a^2b^{-\frac{1}{2}})^{-\frac{1}{2}}$. 56. $(8x^4y^5z^6)^{\frac{1}{3}}$.
57. $(x\sqrt{y})^2$. 58. $(a^2\sqrt[3]{x^2})^6$. 59. $(\sqrt[3]{36x^2})^6$.
60. $\sqrt[3]{a^2\sqrt{a^3}}$. 61. $(\sqrt{(a+x)\sqrt{x}})^4$. 62. $\sqrt[3]{x^2(x+y)^{\frac{1}{2}}}$.

RADICALS.

262. A radical expression is one that contains a radical sign; **a rational** expression is one that can be written without a radical sign or a fractional exponent.

263. A **surd** is a radical expression that cannot be shown to be rational.

264. A surd whose index is 2 is called a quadratic surd; if the index is 3, a cubic surd; and if necessary similar names could be found for radicals of index greater than 3. Equiradical surds are those having the same index.

265. A **mixed surd** is a surd with a coefficient outside the radical sign ; an **entire surd** is one without such a coefficient.

EXERCISE CXXVI.

Express as entire surds :

1. $2\sqrt{53}$. **2.** $3\sqrt{7}$. **3.** $18\sqrt{2}$. **4.** $.01\sqrt{1000}$.

5. $29\sqrt{2}$. **6.** $2\sqrt{29}$. **7.** $111\sqrt{11}$. **8.** $3\sqrt{87}$.

9. $87\sqrt{3}$. **10.** $100\sqrt{.00089}$. **11.** $3\sqrt[3]{9}$. **12.** $9\sqrt[3]{3}$.

13. $11\sqrt[3]{111}$. **14.** $2\sqrt[3]{4}$. **15.** $4\sqrt[3]{2}$. **16.** $2\sqrt[3]{3}$.

17. $3\sqrt[3]{2}$. **18.** $17\sqrt[3]{3}$. **19.** $17\sqrt{3}$. **20.** $2\sqrt[5]{3}$.

21. $3\sqrt[5]{2}$. **22.** $2\sqrt[5]{27}$. **23.** $3\sqrt[5]{16}$. **24.** $13\sqrt{11}$.

25. $\sqrt{4}\sqrt[4]{2}$. **26.** $\sqrt[3]{27}\sqrt[6]{3}$. **27.** $\sqrt{9}\sqrt[3]{2}$. **28.** $\sqrt{2}\sqrt[3]{3}$.

29. $\sqrt[3]{2}\sqrt{3}$. **30.** $\sqrt{3}\sqrt[3]{4}$.

266. A surd is sometimes called an irrational expression. **Similar surds** are those whose irrational factors are the same.

267. A surd term is said to be in its **simplest form** when it is written with **only one radical sign;**

no rational factors in the radical;
no fractions in the radical;
no radical sign in the denominator.

And in general any algebraic expression, to be in its simplest form, must have no fractional or negative exponents.

EXERCISE CXXVII.

Express in the simplest form :

1. $\sqrt{12}$. **2.** $\sqrt{18}$. **3.** $\sqrt{50}$. **4.** $\sqrt{108}$. **5.** $\sqrt{300}$.

6. $\sqrt{448}$. **7.** $\sqrt{726}$. **8.** $\sqrt{1584}$. **9.** $\sqrt{1728}$. **10.** $\sqrt{845}$.

11. $.002\sqrt{500000}$. **12.** $\frac{1}{8}\sqrt{176}$. **13.** $\frac{\sqrt{1682}}{58}$.

14. $.004\sqrt[5]{6250000}$. **15.** $\left(\dfrac{512}{\sqrt[3]{512}}\right)^{-\frac{3}{2}}$. **16.** $12 \div (\sqrt[5]{2187})^{-\frac{1}{3}}$.

17. $\dfrac{a^2+9}{a^2-9} \div (\sqrt{a^3-9a^2+27a-27})^{-2}$. **18.** $4a^3b^{-2}\sqrt[3]{8a^5b^{-6}}$.

19. $275x^{\frac{1}{3}} : (\sqrt[3]{15625x^5})^2$. **20.** $15a^2(x^2-y^2)^{-3} : 3a^{-2}\sqrt{(x-y)^{-3}}$.

Unite similar surds in the following expressions:

21. $\sqrt{8} - 2\sqrt{50} + \sqrt{98}$. **22.** $\sqrt{343} - \sqrt{27} - \sqrt{12}$.

23. $\sqrt{507} + 3\sqrt{12} - 8\sqrt{75} + \sqrt{50}$. **24.** $\sqrt{1000} + \sqrt{360} - \sqrt{20}$.

25. $\sqrt{800} - \sqrt[3]{800} + \sqrt[3]{270} - \sqrt{72}$.

26. $\sqrt{32} + \sqrt[3]{32} + \sqrt[3]{108} - \sqrt{18}$.

27. $\sqrt{48} - \sqrt[3]{81} - \sqrt{675} + \sqrt[3]{648}$.

28. $\sqrt[3]{320} + \sqrt[3]{500} - \sqrt[3]{1372} + \sqrt[3]{2560}$.

29. $\sqrt[4]{6400} + \sqrt{1445} - \sqrt{405}$. **30.** $\sqrt[6]{2916} + \sqrt[3]{432} - \sqrt[3]{250}$.

Distributive Law for Indices.

268. The expression $a^7b^7c^7$ contains 7 factors each of a, b, and c; with one of each in a set, there would be 7 sets of factors,—

$$a^7b^7c^7 \equiv (abc)^7.$$

In general powers of different roots, having the same index, follow this law: **The product of the powers is equal to the power of the product.**

Law VII. $$a^nb^nx^n \equiv (abx)^n.$$

269. This law holds also for quotients:

$$\frac{a^2b^2}{c^2} \equiv \frac{ab}{c} \cdot \frac{ab}{c} \equiv \left(\frac{ab}{c}\right)^2$$

$$\frac{a^nb^n}{c^n} \equiv \left(\frac{ab}{c}\right)^n$$

270. The same law holds for roots:

$\sqrt{a}\sqrt{b} \equiv \sqrt{ab}$; because $\sqrt{a}\sqrt{b}\sqrt{a}\sqrt{b} \equiv \sqrt{a}\sqrt{a}\sqrt{b}\sqrt{b} \equiv ab$.

271. Similarly for all fractional exponents:

$$\left(a^{\frac{p}{q}}b^{\frac{p}{q}}\right)^q \equiv (ab)^p$$

$$\therefore \quad a^{\frac{p}{q}}b^{\frac{p}{q}} \equiv (ab)^{\frac{p}{q}}$$

Hence the law (VII) holds for ALL EXPONENTS, positive or negative, integral or fractional.

272. Radicals with different indices can be reduced to radicals with the same index by expressing them with fractional exponents and reducing the fractional exponents to a common denominator. Thus any two radicals can have their product or quotient written as one radical.

Model B.

$$\sqrt[3]{2}\sqrt[5]{3} = 2^{\frac{1}{3}}3^{\frac{1}{5}} = 2^{\frac{5}{15}}3^{\frac{3}{15}} = \sqrt[15]{2^5 3^3} = \sqrt[15]{(32)(27)} = \sqrt[15]{864}.$$

EXERCISE CXXVIII.

Reduce:

1. $\sqrt{2}\sqrt[3]{2}$. 2. $\sqrt{5}\sqrt[3]{10}$. 3. $\sqrt{6}\sqrt[3]{12}$. 4. $\sqrt{5}\sqrt[3]{2}$.

5. $\sqrt{2}\sqrt[3]{5}$. 6. $\sqrt{5}\sqrt[3]{3}$. 7. $\sqrt{3}\sqrt[3]{5}$. 8. $\sqrt{5}\sqrt[3]{4}$.

9. $\sqrt{10}\sqrt[3]{4}$. 10. $\sqrt{6}\sqrt[3]{4}$. 11. $\sqrt{2}\sqrt[3]{2}\sqrt[4]{2}$.

12. $\sqrt{3}\sqrt[3]{3}\sqrt[4]{3}$. 13. $\sqrt{2}\sqrt[3]{3}\sqrt[4]{4}$. 14. $\sqrt{2}\sqrt[5]{7}$.

15. $\sqrt[3]{2}\sqrt[4]{3}$. 16. $\sqrt{a}\sqrt[3]{a}$. 17. $\sqrt{a}\sqrt[3]{b}$. 18. $\sqrt{a}\sqrt[3]{a^2}$.

19. $\sqrt{a}\sqrt[3]{b^2}$. 20. $\sqrt{ab}\sqrt[3]{a^2b}$.

Rationalizing a Term.

273. Any radical term may be converted into a rational term by multiplying it by a suitable factor. This operation, however, CHANGES THE VALUE OF THE TERM.

$$\sqrt{2}\sqrt{2} = 2$$
$$\sqrt[3]{2}\sqrt[3]{4} = \sqrt[3]{8} = 2$$

If the radical term is in the denominator or in the numerator of a fraction, we may take advantage of this operation WITHOUT CHANGING THE VALUE OF THE FRACTION. Usually it is desirable to have the denominator rational rather than the numerator. Thus $\frac{\sqrt{2}}{\sqrt{7}}$ may be written

$$\frac{\sqrt{2}\sqrt{2}}{\sqrt{7}\sqrt{2}} = \frac{2}{\sqrt{14}} \text{ or } \frac{\sqrt{2}\sqrt{7}}{\sqrt{7}\sqrt{7}} = \frac{\sqrt{14}}{7};$$

but the second form is easier to calculate, 7 being a simpler divisor than $\sqrt{14}$.

EXERCISE CXXIX.

Rationalize divisors:

1. $\frac{1}{\sqrt{2}}$. **2.** $\frac{3}{\sqrt{2}}$. **3.** $\frac{\sqrt{3}}{\sqrt{2}}$. **4.** $\sqrt{\frac{5}{2}}$. **5.** $\frac{1}{\sqrt[3]{4}}$.

6. $\frac{2}{\sqrt[3]{2}}$. **7.** $\frac{3}{\sqrt{3}}$. **8.** $\frac{3}{\sqrt[3]{9}}$. **9.** $\frac{\sqrt[3]{3}}{\sqrt[3]{9}}$. **10.** $\frac{1}{\sqrt[3]{3}}$.

11. $\frac{\sqrt[3]{4}}{3\sqrt[3]{3}}$. **12.** $\frac{\sqrt{3}}{2\sqrt{2}}$. **13.** $\frac{3}{4\sqrt{2}}$. **14.** $\frac{3}{2\sqrt[3]{2}}$. **15.** $\sqrt[3]{\frac{135}{32}}$.

16. $\frac{a}{\sqrt{b}}$. **17.** $\frac{a^2}{\sqrt{ab}}$. **18.** $\sqrt{\frac{a}{b}}$. **19.** $\frac{\sqrt{xy}}{y\sqrt{x}}$. **20.** $\frac{x\sqrt{3}}{\sqrt{3x}}$.

21. $\dfrac{a\sqrt[4]{8x^2}}{2b\sqrt[4]{4x^3}}$. 22 $\dfrac{3a\sqrt{9x^2y}}{x\sqrt[3]{3x^2y}}$. 23. $\dfrac{a^2bc\sqrt{ab^2c}}{\sqrt{abc^2}}$. 24. $\dfrac{\sqrt[3]{32a^5b^4}}{\sqrt[3]{81a^2b^5}}$.

5. $\dfrac{\sqrt[4]{125}}{\sqrt{5x}}$. 26. $\dfrac{\sqrt[3]{9a^2b}}{\sqrt{3ab^2}}$. 27. $\sqrt{15xy} : \sqrt[3]{225x^2}$.

28. $5\sqrt[3]{50a^2b} : 2\sqrt{10ab}$. 29. $\sqrt[5]{125a^{-6}b^{-3}}$.

30. $\sqrt{1000a^3b^{-1} : 200ab}$.

Unite similar surds in the following expressions:

31. $\sqrt{8} - \sqrt{\frac{1}{2}} + \sqrt{\frac{9}{8}} - \sqrt{\frac{1}{8}}$.

32. $\sqrt{\frac{1}{18}} + \sqrt{\frac{1}{50}} - \sqrt{\frac{9}{50}} + \sqrt{\frac{16}{27}}$.

33. $\sqrt{\frac{1}{3}} + 5\frac{1}{3}\sqrt{12} - \sqrt{\frac{25}{12}} + \sqrt{\frac{1}{12}}$.

34. $\sqrt[3]{\frac{1}{32}} + \sqrt{\frac{49}{18}} - \sqrt{\frac{2}{25}} + \sqrt{\frac{9}{8}}$.

35. $\sqrt[3]{\frac{27}{16}} - \sqrt[3]{\frac{1}{2}} + \sqrt[3]{\frac{1}{54}} - 7\sqrt[3]{\frac{.2}{50}}$.

36. $\sqrt{\frac{3x}{4a}} - \sqrt{\frac{a}{3x}} + \sqrt{\frac{ax}{3}}$. 37. $\sqrt[3]{\frac{5}{108x}} + \sqrt[3]{\frac{2}{100x^2}} - \sqrt[3]{\frac{5x}{32}}$.

38. $\sqrt[3]{16x^3y} - \sqrt[3]{\frac{27}{32y^2}} + x^2\sqrt[3]{\frac{343y}{4}}$.

39. $\sqrt{.03x} + \sqrt{\frac{.27}{x}} - \sqrt{\frac{4.32}{x^3}}$.

40. $\sqrt{\frac{4a^3b^2c}{5}} + \sqrt{\frac{9a}{5b^2c}} - \sqrt{\frac{a^5}{20c^3}}$.

Rationalizing Quadratic Surds.

274. Two binomial quadratic surds which consist of the same two terms with opposite signs between them are sometimes called **conjugate surds**. Their product is, by Theorem A, a rational expression. Where one such factor occurs as the denominator of a fraction we may utilize this fact to rationalize the denominator.

Model C.

$$\sqrt{7} : (\sqrt{2} - \sqrt{7}) = \sqrt{7}(\sqrt{2} + \sqrt{7}) : (2 - 7) = (\sqrt{14} + 7) : -5.$$

Model D.

$$\frac{8}{5 + \sqrt{5}} = \frac{8(5 - \sqrt{5})}{25 - 5} = \frac{8}{20}(5 - \sqrt{5}) = \frac{2}{5}(5 - \sqrt{5}).$$

One important advantage of such reductions is evident when we seek the value of the expressions, having given the numerical value of the square roots. Thus

$$\sqrt{5} = 2.236; \quad \frac{8}{5 + \sqrt{5}} = \frac{8}{7.236}; \quad \frac{2}{5})5 - \sqrt{5}) = \frac{2}{5}(2.764);$$

the reduced expression is obviously easier to calculate.

EXERCISE CXXX.

Rationalize the denominators:

1. $\dfrac{3}{\sqrt{2} - 1}$.
2. $\dfrac{1}{2 - \sqrt{3}}$.
3. $\dfrac{2}{3 - \sqrt{3}}$.
4. $\dfrac{2 + \sqrt{2}}{2 - \sqrt{2}}$.
5. $\dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}$.
6. $\dfrac{3 + 2\sqrt{5}}{3 - 5\sqrt{5}}$.
7. $\dfrac{-1 - \sqrt{5}}{1 - \sqrt{2}}$.
8. $\dfrac{3 + \sqrt{2}}{-1 + \sqrt{2}}$.
9. $\dfrac{8 + 4\sqrt{3}}{6 - 2\sqrt{3}}$.
10. $\dfrac{10\sqrt{2} + 2\sqrt{5}}{5 - 2\sqrt{10}}$.

Find the values of the expressions in the preceding exercise, assuming the following approximate values for the roots:

$$\sqrt{2} = 1.414;\quad \sqrt{3} = 1.732;\quad \sqrt{5} = 2.236;\quad \sqrt{10} = 3.1623.$$

275. Two such multiplications are necessary to rationalize denominators which are quadratic surds of more than two terms.

EXERCISE CXXXI.

Rationalize denominators:

1. $\dfrac{3 + \sqrt{6a} + 2a^2}{3 - \sqrt{6a} + 2a^2}$.
2. $\dfrac{\sqrt{6} - \sqrt{10} + \sqrt{15}}{\sqrt{6} + \sqrt{10} - \sqrt{15}}$.
3. $\dfrac{3 + \sqrt{2}}{1 + \sqrt{2} + \sqrt{3}}$.
4. $\dfrac{2 - \sqrt{5}}{\sqrt{5} - \sqrt{2} + 3}$.
5. $\dfrac{1}{\sqrt{2} + \sqrt{3} + \sqrt{6}}$.

Other Rationalizing Factors.

276. Rationalizing factors can also be obtained for binomial surds that are not quadratic; but their complexity often makes it undesirable to seek for them.

Model E. $\sqrt[3]{2} - \sqrt[3]{5}$ suggests

$$a^3 - b^3 \equiv (a - b)(a^2 + ab + b^2);\quad \text{so that}$$

$$(\sqrt[3]{2} - \sqrt[3]{5})(\sqrt[3]{4} + \sqrt[3]{10} + \sqrt[3]{25}) = 2 - 5 = -3.$$

Model F. $2 + \sqrt[4]{3y}$ suggests

$$(a + b)(a^3 - a^2b - ab^2 + b^3) \equiv a^4 - b^4;\ \text{so that}$$

$$(2 + \sqrt[4]{3y})(8 - 4\sqrt[4]{3y} - 2\sqrt[4]{9y^2} + \sqrt[4]{27y^3}) \equiv 16 + 3y.$$

EXERCISE CXXXII.

Find rationalizing factors for:

1. $1 - \sqrt[3]{2}$.
2. $2 + \sqrt[3]{3}$.
3. $\sqrt[3]{2} - \sqrt[3]{3}$.
4. $\sqrt[4]{2} - 1$.
5. $\sqrt[4]{2} - \sqrt[4]{8}$.

IMAGINARIES.

277. Imaginary quantities present an important exception to the algebra of ordinary radicals, in this respect:

$\sqrt{-2}$ must by definition give -2 when multiplied by itself; law (VII) can therefore not apply to imaginaries, because by its application we should have

$$(-2)^{\frac{1}{2}}(-2)^{\frac{1}{2}} = [(-2)(-2)]^{\frac{1}{2}} = (4)^{\frac{1}{2}} = \sqrt{4}.$$

This would allow $\sqrt{-2}$ to be equal to $\sqrt{2}$.

278. If we allow i^2 to stand for -1 and i for $\sqrt{-1}$, as in a previous chapter, we shall have $\sqrt{-a} = \sqrt{i^2a} = i\sqrt{a}$;

then $\quad (\sqrt{-a})(\sqrt{-a}) = (i\sqrt{a})(i\sqrt{a}) = i^2a = -a.$

The symbol i can be handled according to the ordinary laws of algebra and subject to the definition

$$i^2 = -1.$$

It follows from this that $i^3 = -i$; $i^4 = 1$; and in the expression i^n we may subtract from n any multiple of 4 without altering its value.

279. An expression involving the square root of a negative number is called an imaginary (or a complex) number; and i is called **the imaginary unit.** All imaginary expressions may be reduced to the algebra of ordinary radicals by substituting $i = \sqrt{-1}$ as a coefficient of the radical; and of course no term containing i can be regarded as similar to a term not containing i.

EXERCISE CXXXIII.

Unite similar surds:

1. $\sqrt{-8}+\sqrt{-72}-\sqrt{-18}$.
2. $\sqrt{-27}+\sqrt{-75}-\sqrt{-108}$.
3. $\sqrt{-72}-\sqrt{-243}+\sqrt{-338}-\sqrt{-1200}$.
4. $3\sqrt{-8}+8\sqrt{-3}-\sqrt{6}(\sqrt{-50}-\sqrt{-12})$.
5. $3\sqrt{-32}-2\sqrt{-48}-\sqrt{-12}(\sqrt{6}+2)$.
6. $\sqrt{-200}-\sqrt[3]{\frac{27}{5}}+\sqrt[3]{5}(\sqrt[3]{-40}-\sqrt[3]{135})$.
7. $\sqrt{-200}-\sqrt{3}\left(\sqrt{\frac{27}{2}}-\sqrt{-\frac{27}{2}}\right)+\sqrt{200}$.
8. $\sqrt{-500}-\sqrt{2}\left(\sqrt{\frac{49}{90}}+\sqrt{-\frac{121}{40}}\right)-\sqrt{\frac{-4}{5}}$.
9. $\sqrt{-75}-\sqrt{-27}+\sqrt{48}-\sqrt{-108}$.
10. $\sqrt{1250}-\sqrt{-50}+\sqrt{150}-\sqrt{-450}-\sqrt{24}$.

Simplify:

11. $(\sqrt{2}+\sqrt{-1})^2$.
12. $(\sqrt[3]{2}+\sqrt{-2})^3$.
13. $(\sqrt{2}+\sqrt{-2})^2$.
14. $(\sqrt{5}+\sqrt{-2})^2$.
15. $(2\sqrt{3}-3\sqrt{-2})^2$.
16. $(\sqrt{-12}+\sqrt{-18})^3$.
17. $(\sqrt{50}+\sqrt{-27})(\sqrt{48}-\sqrt{-72})$.
18. $(\sqrt{8}+\sqrt{-75}-\sqrt{200})(\sqrt{-32}-\sqrt{-675})$.
19. $\left(\sqrt{\frac{1}{2}}-\sqrt{-\frac{1}{3}}\right)(\sqrt{-2}+6\sqrt{-3})$.
20. $\left(\sqrt{\frac{25}{-18}}+\sqrt{\frac{-4}{3}}\right)(6\sqrt{6}-18\sqrt{2})$.

Square Root of a Binomial Quadratic Surd.

280. From the identities

$$(\sqrt{x}+\sqrt{y})^2 \equiv x+y+2\sqrt{xy}$$
$$(\sqrt{x}-\sqrt{y})^2 \equiv x+y-2\sqrt{xy}$$

we conclude :

The square of a binomial quadratic surd is a binomial surd with one term rational.

And also :

The square root of a binomial quadratic surd of which one term is rational is the sum (or the difference) of the square roots of two numbers such that:

their sum equals the rational term;

their product equals the square of half the surd term; in the given binomial surd.

EXERCISE CXXXIV.

Find the square root of :

1. $3+2\sqrt{2}$.
2. $7-4\sqrt{3}$.
3. $5-2\sqrt{6}$.
4. $6-2\sqrt{5}$.
5. $7+4\sqrt{3}$.
6. $19-6\sqrt{10}$.
7. $33-20\sqrt{2}$.
8. $35+12\sqrt{6}$.
9. $23-8\sqrt{7}$.
10. $310-66\sqrt{21}$.
11. $9\frac{1}{2}-3\sqrt{2}$.
12. $6\frac{1}{2}-\sqrt{30}$.
13. $10\frac{1}{12}-\frac{1}{3}\sqrt{30}$.
14. $70+40\sqrt{3}$.
15. $1338-72\sqrt{42}$.
16. $3.31-.22\sqrt{210}$.
17. $12.55-.7\sqrt{30}$.
18. $7\frac{7}{9}-\frac{14}{9}\sqrt{21}$.
19. $\frac{751}{80}-\frac{6}{5}\sqrt{55}$.
20. $64.8-1.6\sqrt{80}$.

281. When both terms of the given expression are surd, the square root cannot be found without removing a surd factor, the square root of which must be found separately.

EXERCISE CXXXV.

Find the square root of:

1. $16\sqrt{3} - 2\sqrt{165}$.
2. $13\sqrt{7} + 2\sqrt{210}$.
3. $5\sqrt{7} - 7\sqrt{3}$.
4. $8\sqrt{2} + 2\sqrt{14}$.
5. $7\sqrt{2} - 4\sqrt{5}$.

EXERCISE CXXXVI.

Substitute, in the following expressions,

$$a = \sqrt{3};\ b = 3\sqrt{2};\ c = \sqrt{\frac{5}{2}};$$

$$x = \sqrt{q};\ y = k\sqrt{q};\ z = 3\sqrt{2k};$$

and simplify the results:

1. $\dfrac{a^2b}{b^2c}$.
2. $\dfrac{3ab}{x^2y}$.
3. $(a - b)^2$.
4. $(a - x)^2$.
5. $\dfrac{a^2b}{27}$.
6. $45a^8y^2 - 120a^7y^3$.
7. $55x^9z^9y^2 - 165x^8z^8y^3$.
8. $\dfrac{(a-y)^2}{x^3 - z^3}$.
9. $\dfrac{b^2 - x^3}{y^2 - a^3}$.
10. $\dfrac{x^4 - z^4}{(c^2 - y^2)^2}$.

In the same expressions substitute (and simplify as before):

$$a = \sqrt[3]{4};\ b = 5\sqrt[4]{8};\ c = \frac{1}{30}\sqrt{5};$$

$$x = q^{\frac{2}{3}};\ y = kq^{-\frac{3}{4}};\ z = 30k^{-\frac{1}{2}}h^{\frac{1}{3}}.$$

In the same expressions substitute (and simplify as before):

$$a = \frac{9a}{\sqrt{b}};\ b = \frac{2b}{\sqrt{a}};\ c = 2\sqrt{-3};$$

$$x = \frac{6a}{7b\sqrt{b}};\ y = \frac{b}{\sqrt{3a}};\ z = 6\sqrt{-5}.$$

Square Roots of Complex Binomials.

282. In finding square roots of binomial surds we often meet imaginary expressions.

EXERCISE CXXXVII.

Find the square root of:

1. $2\sqrt{-2}-1.$
2. $1+2\sqrt{-2}.$
3. $2+4\sqrt{-2}.$
4. $6-6\sqrt{-3}.$
5. $26+4\sqrt{-30}.$
6. $118+2\sqrt{-3}.$
7. $-4-6\sqrt{-5}.$
8. $-19-10\sqrt{-6}.$
9. $51+14\sqrt{-10}.$
10. $8+2\sqrt{-65}.$

Radical Equations.

283. Some radical equations can be solved by taking the radical itself as the unknown letter (and in that case the problem is much simplified by letting z stand for the radical).

Model G.

① $\frac{7}{4}-\frac{2\sqrt{x}-5}{\sqrt{x}+5}=\frac{3\sqrt{x}-7}{2\sqrt{x}}$

Let $z=\sqrt{x}.$

② $\frac{7}{4}-\frac{2z-5}{z+5}=\frac{3z-7}{2z}$

③ $7z^2+35z-8z^2+20z=6z^2+16z-70$ ② $\times 4z(z+5)$

④ $7z^2-39z-70=0$ ③ $+z^2-55z$

⑤ $(7z+10)(z-7)=0$ ④ factored

Whence $z=-\frac{10}{7};\ z=7.$

But $z=\sqrt{x}$; hence $x=\frac{100}{49};\ x=49.$

Notice that $x = \frac{100}{49}$ will not satisfy the given equation unless the radical is taken with a minus sign.

Model H.

(1) $\sqrt{3x+4} + \sqrt{5x+1} + \sqrt{18x-5} = 0$

(2) $\sqrt{3x+4} + \sqrt{5x+1} = -\sqrt{18x-5}$ (1) $-\sqrt{18x-5}$

(3) $3x+4+2\sqrt{(5x+1)(3x+4)}+5x+1 = 18x-5$ (2)^2

(4) $2\sqrt{(15x^2+23x+4)} = 10x-10$ (3) $-8x-5$

(5) $\sqrt{15x^2+23x+4} = 5x-5$ (4) $\div 2$

(6) $15x^2+23x+4 = 25x^2-50x+25$ (5)^2

(7) $10x^2-73x+21 = 0$ (6)$-15x^2-12x-4$

(8) $(x-7)(10x-3) = 0$ (7) factored

Whence $x = 7$; $x = \frac{3}{10}$.

Notice that (1) if squared would give three double cross products; (2) squared gives only one; and a similar reason holds for the change from (3) to (4). Again, we must notice that the value $x = 7$ satisfies (1) only if we take the sign of $\sqrt{18x-5}$ minus; and the value $x = .3$ satisfies (1) only if $\sqrt{5x+1}$ and $\sqrt{18x-5}$ are both minus.

EXERCISE CXXXVIII.

Solve the equations:

1. $8\sqrt{x} + \frac{4\sqrt{x}-1}{2-\sqrt{x}} = 5\sqrt{x} + \frac{3x+9}{1+\sqrt{x}}.$

2. $\frac{\sqrt{x}+4}{\sqrt{x}-4} - \frac{\sqrt{x}-4}{\sqrt{x}+4} = \frac{24}{5}.$

3. $\frac{3\sqrt{x}+2}{1+\sqrt{x}} = 2\,\frac{3+2\sqrt{x}}{5+2\sqrt{x}}.$

4. $\frac{4\sqrt{x}}{10+\sqrt{x}} = \frac{5-2\sqrt{x}}{2\sqrt{x}+4} + \frac{21}{8}.$

5. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}-\dfrac{4}{3}=\dfrac{3\sqrt{x}+1}{2(1+\sqrt{x})}.$

6. $\sqrt{4x-3}+\sqrt{x+2}=\sqrt{9x+1}.$

7. $\sqrt{4x+4}+\sqrt{x-4}=\sqrt{8x}.$

8. $\sqrt{11x+11}+\sqrt{x-1}=\sqrt{20x-4}.$

9. $\sqrt{2x+3}+\sqrt{x-2}=\sqrt{5x+1}.$

10. $\sqrt{4x-8}+\sqrt{x-2}=\sqrt{x}.$ 11. $\sqrt{x+1}+\sqrt{x}=\dfrac{2}{\sqrt{x}}.$

12. $\sqrt{x+4}+\sqrt{x}=\dfrac{6}{\sqrt{x}}$ 13. $\sqrt{4x-6}+\sqrt{x}=\dfrac{3}{\sqrt{x}}.$

14. $\sqrt{6x+30}+\sqrt{x}=\dfrac{4}{\sqrt{x}}.$ 15. $\sqrt{17x+28}+\sqrt{x}=\dfrac{7}{\sqrt{x}}.$

16. $\sqrt{x-2}=\frac{1}{3}\sqrt{x}.$ 17. $\sqrt{x-2}=\sqrt{5x+1}-\sqrt{2x+3}.$

18. $\sqrt{x-2}=\sqrt{9x-2}-\sqrt{5x+1}.$

19. $3\sqrt{x-1}-\sqrt{4x+9}=\sqrt{x-5}.$

20. $\sqrt{7x+4}-\sqrt{2x+9}=\sqrt{x+5}.$

21. $\dfrac{6}{1-\sqrt{x}}-\dfrac{2}{\sqrt{x}}=3.$ 22. $\dfrac{3}{\sqrt{x}}+\dfrac{1}{5}=\dfrac{2\sqrt{x}+1}{x}.$

23. $\sqrt{2x}-\sqrt{x-34}=\sqrt{x-62}.$

24. $\sqrt{x+1}+\frac{1}{2}\sqrt{x-4}=\sqrt{2x}.$

25. $\sqrt{15x-5}+\sqrt{x}=\dfrac{2}{\sqrt{x}}.$ 26. $2\sqrt{x-8}+\sqrt{x}=\dfrac{3}{\sqrt{x}}.$

27. $\sqrt{5x+1}+\sqrt{x-2}=\sqrt{9x-2}.$

28. $\sqrt{9x-9}-\sqrt{4x+9}=\sqrt{x-5}.$

29. $\sqrt{9x+1}-\sqrt{3x+3}=\sqrt{x+5}.$

30. $\sqrt{7x+4}-\sqrt{2x+9}=\sqrt{x+5}.$

31 $\sqrt{\dfrac{x-2}{x}}+1=\dfrac{6}{x}.$ 32. $\sqrt{x+5}=\dfrac{3}{\sqrt{x}}-\sqrt{x}.$

33. $\sqrt{x+6}+\sqrt{x-1}=\dfrac{4}{\sqrt{x-1}}$.

34. $\dfrac{1}{2+\sqrt{x}}=\dfrac{1}{2\sqrt{x}-4}-\dfrac{3}{10}$.

35. $\dfrac{2+\sqrt{x}}{2-\sqrt{x}}-\dfrac{1-\sqrt{x}}{1+\sqrt{x}}=\dfrac{9}{5}$.

36. $\dfrac{1}{\sqrt{x}}-\dfrac{1}{1+\sqrt{x}}=\dfrac{3}{10(2+\sqrt{x})}$.

37. $\sqrt{\dfrac{9x+1}{3}}-\sqrt{x+1}=\sqrt{\dfrac{x+5}{3}}$.

38. $\sqrt{x+1}+\sqrt{\dfrac{x-1}{11}}=\dfrac{2}{11}\sqrt{5x-1}$.

39. $\sqrt{8x+1}+\sqrt{5x+4}+\sqrt{x-19}=0$.

40. $\sqrt{2x}-\sqrt{x-34}=\sqrt{x-62}$.

41. $\sqrt{8x+1}-\sqrt{5x+4}=\sqrt{x-19}$.

42. $\dfrac{3\sqrt{x}-2}{2\sqrt{x}-3}-\dfrac{2\sqrt{x}-3}{3\sqrt{x}-2}=\dfrac{15}{4}$.

43. $5\sqrt{x}-\dfrac{15-\sqrt{x}}{4+\sqrt{x}}=15$.

44. $\sqrt{4x+3}-\sqrt{x+2}-\sqrt{9x+1}=0$.

45. $\dfrac{45}{5-\sqrt{x}}=75-\dfrac{15\sqrt{x}}{7}$.

46. $\sqrt{x+2}+\sqrt{x-1}=\dfrac{2}{\sqrt{x+2}}$.

47. $\sqrt{x+5}+\sqrt{x}=\dfrac{3}{\sqrt{x+5}}$.

48. $\sqrt{x+8}+1=\dfrac{5}{\sqrt{x^2+8x}}$.

49. $\sqrt{2x-79}=\sqrt{x}+\dfrac{11}{\sqrt{2x-79}}$.

50. $\dfrac{3}{\sqrt{x}+2}=\dfrac{13}{5}-\dfrac{2}{7-2\sqrt{x}}$.

51. $\sqrt{\dfrac{x}{x+20}}=\dfrac{14}{x+20}-\sqrt{x}$.

Equations Solved Like Quadratics.

284. In the following examples the expression under the radical sign, or some multiple of it, can be put together out of the terms that form the rest of the equation, in such a way that if z be substituted for the radical the equation may be treated as a quadratic in z. After solving this the radical is again substituted for z, and that equation solved for x.

Since the given equation would be of the fourth degree if cleared of radicals, four values of x are expected.

Model I.

(1) $\dfrac{1-\sqrt{2x^2-3x-5}}{x}=\dfrac{3-2x}{9}$

(2)	$9-9\sqrt{2x^2-3x-5}=3x-2x^2$	(1) $\times\, 9x$
(3)	$2x^2-3x+9-9\sqrt{2x^2-3x-5}=0$	(2) $-3x+2x^2$
(4)	$2x^2-3x-5-9\sqrt{2x^2-3x-5}+14=0$	Same as (3)
(5)	$z^2-9z+14=0$	[In this equation $z=\sqrt{2x^2-3x-5}$]
(6)	$(z-7)(z-2)=0$	(5) factored
(7)	$z-7=0;\ z-2=0$	(6) Ax. A
(8)	$z=7;\ z=2$	from (7)
(9)	$\sqrt{2x^2-3x-5}=7;\ 2x^2-3x-5=49$	from (8)
(10)	$2x^2-3x-54=0$	(9) $-\ 49$

(11) $(2x + 9)(x - 6) = 0$ (10) factored

(12) $x = -\frac{9}{2};\ x = 6$

(13) $2x^2 - 3x - 5 = 4$ from (8)

(14) $2x^2 - 3x - 9 = 0$ (13) $- 4$

(15) $(2x + 3)(x - 3) = 0$ (14) factored

(16) $x = -\frac{3}{2};\ x = 3$

Ans. $x = 3;\ x = 6;\ x = -\frac{9}{2};\ x = -\frac{3}{2}$.

EXERCISE CXXXIX.

In the same way solve:

1. $4x^2 - 13x = 18\sqrt{2x^2 - 7x} - 28 + x$.
2. $x + 2x^2 + 8\sqrt{x^2 - 3x - 9} = 7x + 12$.
3. $x^2 - 4x - 34\sqrt{x^2 - 5x - 2} = x - 62$.
4. $\dfrac{x - 7}{6} = \dfrac{\sqrt{x^2 - 4x - 20} - 1}{x + 3}$.
5. $4 + \sqrt{3x^2 - 12x - 71} = \frac{3}{13}(x^2 - 4x + 7)$.
6. $8x^2 + x - 11 + \sqrt{8x^2 + x - 5} = 0$.
7. $3x^2 - 11\sqrt{3x^2 - x + 23} = x - 53$.
8. $8x^2 + 6x - 5(1 + \sqrt{4x^2 + 3x - 1}) = 0$.
9. $6(x^2 + 2x + 8) = 13\sqrt{3x^2 + 6x + 19} - 10$.
10. $\sqrt{7x^2 - x + 30} = \frac{3}{8}(7x^2 - x + 10)$.

EXERCISE CXL.

In order to find the value of x in the following equations, it may be necessary to extract the square root of a binomial surd:

1. $x^4 - 10x^2 + 1 = 0.$
2. $x^4 - 22x^2 + 25 = 0.$
3. $x^4 - 60x^2 + 36 = 0.$
4. $x^4 + 1 = 14x^2.$
5. $38x^2 - x^4 = 215.$
6. $1 = 6x^2 - x^4.$
7. $\frac{x^2}{2} = \sqrt{5x^2 + 7}.$
8. $\frac{x^4 + 8}{x^2 - 5} = 28.$
9. $x^2 - 15 = 2x\sqrt{3}.$
10. $\frac{x^4 + 100}{4} = 7x^2.$

The following equations yield surd answers, which must be reduced to their lowest terms:

11. $x^2 - 30x + 9 = 0.$
12. $x^2 + 10 = 16x.$
13. $x^2 + 20x = 50.$
14. $x^2 = 17 - 32x.$
15. $x^2 + 28x + 71 = 0.$
16. $61 = 26x - x^2.$
17. $\frac{x^3 - x^2}{5 - 5x} = 4x + 5.$
18. $x^2 + 32x + 106 = 0.$
19. $-x = 2 + \frac{x^2}{18}.$
20. $\frac{x^2 + 3}{11} = 2x - 2.$
21. $\frac{x}{6} = \sqrt{x - 4}.$
22. $\frac{x^2}{6} + 10 = 6(x - 6).$
23. $\frac{x^2}{6} - 10 = 6(x + 6).$
24. $(3x - 8)^2 = 48x.$
25. $x = 8 + \frac{1}{2}\sqrt{44x + 3}.$
26. $7x^2 : (35x - 2) = 8 : 7.$
27. $16(x^2 + 4) = 120x + 1.$
28. $x^2 + 4.8x + 1.84 = 0.$
29. $x^2 + 11 = 17.5x.$
30. $.16 = \frac{1}{x} - \frac{.0625}{x^2}.$

Express the answers to the last 20 examples decimally, assuming

$$\sqrt{2} = 1.414; \quad \sqrt{3} = 1.732;$$
$$\sqrt{5} = 2.236; \quad \sqrt{6} = 2.4495.$$

ROOTS.

Square Roots of Numbers.

285. In finding the square root of 383161, we first see that the result is larger than 600 (since 383161 > 360000) and subtract the square of that; then, concluding that the remainder of the square root is not less than 10, we subtract the REST OF THE SQUARE of 600 + 10.

$$
\begin{array}{r|r}
383161 & 619 \\
36 & \\
\hline
231 & 121 \\
121 & 1 \\
\hline
11061 & 1229 \\
11061 & 9 \\
\hline
0 &
\end{array}
$$

Representing 600 by a and 10 by b, we see that we have subtracted first a^2, then $(2a + b)b$, in all $(a + b)^2$ or 610^2. Next we decide that the remainder of the result is not less than 9, and so subtract the REST OF THE SQUARE of 610 + 9.

If this process is not concluded by a zero remainder, it can be continued indefinitely.

Square Roots of Algebraic Expressions.

286. The square roots of algebraic polynomials are found in the same way.

Model J.

$$
\begin{array}{r|l}
a^4 + 6a^3b - a^2b^2 - 30ab^3 + 25b^4 & a^2 + 3ab - 5b^2 \quad \textit{Ans.} \\
a^4 & \\
\hline
+ 6a^3b - a^2b^2 - 30ab^3 + 25b^4 & 2a^2 + 3ab \\
+ 6a^3b + 9a^2b^2 & + 3ab \\
\hline
- 10a^2b^2 - 30ab^3 + 25b^4 & 2a^2 + 6ab - 5b^2 \\
- 10a^2b^2 - 30ab^3 + 25b^4 & - 5b^2 \\
\hline
0 &
\end{array}
$$

A large part of the work of this example consists in rewriting the same terms, as remainders in successive subtractions. The work may be contracted as follows:

Model J.

$$
\begin{array}{c|l}
\cancel{a^4} + \cancel{6a^3b} - a^2b^2 - \cancel{30ab^3} + \cancel{25b^4} & a^2 + 3ab - 5b^2 \\
9a^2b^2 & 2a^2 + 3ab \\
\hline
- \cancel{10a^2b^2} & 2a^2 + 6ab - 5b^2
\end{array}
$$

EXERCISE CXLI.

Find the square roots of:

1. $9x^4 + 42x^3 + 37x^2 - 28x + 4.$
2. $4x^6 - 4x^4 + 4x^3 + x^2 - 2x + 1.$
3. $20x^3 - 26x^2 - 12x + 9 + 25x^4.$
4. $6x^5 + 26x^3 - 11x^4 + 9x^6 + 14x^2 - 20x + 25.$
5. $49x^4 + 56x^5 - 24x^3 - 42x^2 + 9 + 16x^6.$

Series for Square Root.

287. With algebraic expressions, as well as with numbers, if there is no exact square root the process may be continued indefinitely. This leads to a succession of algebraic terms which will always be incomplete, however far extended.

Model K.

$1 - x$	$1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{16} - \frac{5x^4}{128} -$
1	
$-x$	$2 - \frac{x}{2}$
$-x + \frac{x^2}{4}$	$-\frac{x}{2}$
$-\frac{x^2}{4}$	$2 - x - \frac{x^2}{8}$
$-\frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{64}$	$-\frac{x^2}{8}$
$-\frac{x^3}{8} - \frac{x^4}{64}$	$2 - x - \frac{x^2}{4} - \frac{x^3}{16}$
$-\frac{x^3}{8} + \frac{x^4}{16} + \frac{x^5}{128} + \frac{x^6}{256}$	$-\frac{x^3}{16}$
$-\frac{5x^4}{64} - \frac{x^5}{128} - \frac{x^6}{256}$	$2 - x - \frac{x^2}{4} - \frac{x^3}{8} - \frac{5x^4}{128}$
$-\frac{5x^4}{64} + \frac{5x^5}{128} + \frac{5x^6}{512} + \frac{5x^7}{1024} + \frac{25x^8}{16384}$	$-\frac{5x^4}{128}$

EXERCISE CXLII.

Find series for:

1. $\sqrt{1+2x}$.
2. $\sqrt{4+5x}$.
3. $\sqrt{1+2x-3x^2}$.
4. $\sqrt{25+x}$.
5. $\sqrt{9+8x}$.
6. $\sqrt{x^2+1}$.

Cube Roots of Numbers.

288. The process for cube root is closely analogous to that for square root, and is suggested by the identity

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3.$$

Model I. $\sqrt[3]{584277056}$.

First we decide that the cube root is not less than 800; and subtract the cube of that:

$$\begin{array}{r|l} 584\,277\,056 & 800 = a \\ 512\,\mathit{000\,000} & \\ \hline 72\,277\,\mathit{056} & \end{array}$$

Call 800 a and the next part of the root that we expect to find b. Since we have subtracted a^3, the rest of the cube of $(a+b)$ is

$$3a^2b + 3ab^2 + b^3 \equiv b(3a^2 + 3ab + b^2);$$

and the remainder, 72,277,056, will be larger than that.

Now if we knew the value of $3a^2 + 3ab + b^2$ we could find b at once by dividing. Although we do not know it, we are helped in guessing at b by the fact that $3a^2$ is much the larger part of the divisor.

In this case $3a^2 = 1920000$, and we conclude that b is

not less than 30; on the supposition that $a = 800$ and $b = 30$ we construct $b(3a^2 + 3ab + b^2)$ and subtract.

584 277 056	8*00* $= a$
	3*0* $= b$
512 *000 000*	
72 277 *056*	1920*000* $= 3a^2$
	720*00* $= 3ab$
	9*00* $= b^2$
59 787 *000*	19929*00* $= 3a^2 + 3ab + b^2$
12 490 056	

We have now subtracted the cube of 830; we may call 830 a, construct $3a^2$, and proceed as before, until there is no remainder, or until enough figures of the root are obtained.

Model L.

584 277 056	$\left.\begin{array}{r} 8\mathit{00} = a \\ 3\mathit{0} = b \end{array}\right\} = a$	*Ans.* 836.
	6 $= b$	
512 *000 000*		
72 277 *056*	1920*000* $= 3a^2$	$a =$ 80*0*
	720*00* $= 3ab$	$b =$ 3*0*
	9*00* $= b^2$	
59 787 *000*	19929*00* $= 3a^2 + 3ab + b^2$	
12 490 056	2066700 $= 3a^2$	$a =$ 830
	14940 $= 3ab$	$b =$ 6
	36 $= b^2$	
12 490 056	2081676 $= 3a^2 + 3ab + b^2$	

In practice the figures in italics in the preceding problem are omitted, so that at each successive step we seem to be dealing with tens and units. And it often happens that

the first guess for b is a wild one,— particularly if the first figure of the root is small.

Model M. $\sqrt[3]{56623104}$.

56 623 104	384	
27		
29 623	$2700 = 3a^2$	$a = 30$
	$720 = 3ab$	$b = 8$
	$64 = b^2$	
27 872	$3484 = 3a^2 + 3ab + b^2$	
1 751 104	$433200 = 3a^2$	$a = 380$
	$4560 = 3ab$	$b = 4$
	$16 = b^2$	
1 751 104	$437776 = 3a^2 + 3ab + b^2$	

Model N. $\sqrt[3]{29.23}$.

29.230 000 000	3.0804	
27		
2 230	$2700 = 3a^2$	$a = 30$
		$b = 0$
2 230 000	$270000 = 3a^2$	$a = 300$
	$7200 = 3ab$	$b = 8$
	$64 = b^2$	
2 218 112	$277264 = 3a^2 + 3ab + b^2$	
11 888 000	$28459200 = 3a^2$	$a = 3080$
		$b = 0$
11 888 000 000	$2845920000 = 3a^2$	$a = 30800$
	$3696000 = 3ab$	$b = 4$
	$16 = b^2$	
11 398 464 064	$2849616016 = 3a^2 + 3ab + b^2$	
489 535 996		

EXERCISE CXLIII.

Find the cube roots of:

1. 51895117. 2. 63044792. 3. 9181846584.
4. 7587307125. 5. 6946005312. 6. 2700.
7. 54. 8. 100. 9. 2. 10. 20000.

Cube Roots of Algebraic Expressions.

289. For cube roots of algebraic expressions the work is just the same in principle, but the result can generally be found by inspection—unless the polynomial is very long indeed—if one is sure that the given expression is a perfect cube. For the first term and the last term of the cube root are respectively the cube roots of first and last straight products; and the terms next to each of those in the root can be seen by dividing the next term in the given expression by the "trial divisor" ($3a^2$).

Model 0.—The first and last terms of

$$8a^9+12a^8-18a^7+13a^6+54a^5-51a^4-10a^3+63a^2-54a+27$$

are respectively $2a^3$ and $+3$; starting from the first term, $3a^2$ would be $3(2a^3)^2 \equiv 12a^6$; so b would be $12a^8 \div 12a^6 \equiv a^2$; rearranging the terms in reverse order, so as to start with the last term, $3a^2 = 3(9) = 27$; then b would be $-54a \div 27 \equiv -2a$. The root would be $2a^3 + a^2 - 2a + 3$. To verify this result it is easier to go through the motions of finding a cube root than to multiply out the third power.

Model P.

$\cancel{8a^9}+12a^8-18a^7+13a^6+54a^5$ $-51a^4+10a^3+63a^2-54a+27$	$2a^3+a^2-2a+3$
$\cancel{8a^9}$	$12a^6=3a^2 \qquad a=2a^3$
$12a^8+6a^7+a^6$	$6a^5=3ab \qquad b=a^2$ $a^4=b^2$
$-24a^7+12a^6+54a^5-51a^4+10a^3$ $+63a^2-54a+27$	$12a^6+12a^5+3a^4=3a^2$ $-12a^4-6a^3=3ab \qquad a=2a^3+a^2$ $4a^2=b^2 \qquad b=-2a$
$-24a^7-24a^6+18a^5+12a^4-8a^3$	
$36a^6+36a^5-63a^4+18a^3+63a^2$ $-54a+27$	$12a^6+12a^5-21a^4-12a^3+12a^2=3a^2$ $18a^3+9a^2-18a=3ab$ $9=b^2$
$36a^6+36a^5-63a^4+18a^3+63a^2$ $-54a+27$	$[a=2a^3+a^2-2a]$ $[b=3]$

EXERCISE CXLIV.

Find the cube roots of:

1. $x^6+96x-40x^3+6x^5-64.$
2. $x^6+12x^5y+60x^4y^2+160x^3y^3+240x^2y^4+192xy^5+64y^6.$
3. $27x^9-54x^8+63x^7-71x^6+57x^5-36x^4+22x^3-9x^2+3x-1.$
4. $8a^6-12a^5+18a^4-13a^3+9a^2-3a+1.$
5. $x^{12}-3x^{11}y+3x^{10}y^2+2x^9y^3-9x^8y^4+9x^7y^5-9x^5y^7+9x^4y^8-2x^3y^9-3x^2y^{10}+3xy^{11}-y^{12}.$
6. $328509x^6-480861x^4yz+283383x^2y^2z^2-50653y^3z^3.$
7. $8x^{12}-12x^{11}+42x^{10}-61x^9+99x^8-117x^7+126x^6-108x^5+81x^4-47x^3+21x^2-6x+1.$

*8. $\frac{1}{8}a^2-\frac{1}{8}ab^5+\frac{3}{2}a^{\frac{4}{3}}b^{\frac{5}{3}}+6a^{\frac{2}{3}}b^{\frac{10}{3}}+8b^{\frac{15}{3}}-\frac{3}{8}a^{\frac{5}{3}}b^{\frac{5}{3}}-12ab^{\frac{10}{3}}-6a^{\frac{1}{3}}b^5+\frac{3}{8}a^{\frac{4}{3}}b^{\frac{10}{3}}+\frac{3}{2}a^{\frac{2}{3}}b^5.$

9. $16x^9\sqrt{2}-72x^6y\sqrt{6}+324x^3y^2\sqrt{2}-162y^3\sqrt{6}.$
10. $x^8-3x^5y^2\sqrt[3]{xy^2}+3x^2y^5\sqrt[3]{x^2y}-y^8.$

* Rearrange.

Series for Cube Root.

290. As in square root, if the given expression is not a perfect power, the result will be an endless series of terms.

Model Q. $\sqrt[3]{1-x}.$

$$
\begin{array}{l|l}
1 - x & 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \dots \\
1 & \\
\hline
-x & 3 = 3a^2 \qquad a = 1 \\
 & -x = 3ab \qquad b = -\frac{x}{3} \\
-x + \frac{x^2}{3} - \frac{x^3}{27} & \frac{x^2}{9} = b^2 \\
\hline
-\frac{x^2}{3} + \frac{x^3}{27} & 3 - 2x + \frac{x^2}{3} = 3a^2 \qquad a = 1 - \frac{x}{3} \\
 & -\frac{x^2}{3} + \frac{x^3}{9} \quad = 3ab \quad b = -\frac{x^2}{9} \\
-\frac{x^2}{3} + \frac{2x^3}{9} - \frac{x^5}{81} - \frac{x^6}{729} & \frac{x^4}{81} = b^2 \\
\hline
-\frac{5x^3}{27} + \frac{x^5}{81} + \frac{x^6}{729} & 3 - 2x - \frac{x^2}{3} + \frac{2x^3}{9} + \frac{x^4}{27}
\end{array}
$$

EXERCISE CXLV.

Find series for :

1. $\sqrt[3]{1+x}.$
2. $\sqrt[3]{1+2x+x^2}.$
3. $\sqrt[3]{8-12x}.$
4. $\sqrt[3]{1-x^3}.$
5. $\sqrt[3]{1-x+x^2}.$

Fourth and Higher Roots.

291. By utilizing in the same way the identities

$$(a+b)^4 \equiv a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 \equiv a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

and so on,

it would be theoretically possible to construct a rule, analogous to those for square and cube root, which would serve to find any desired root. Practically, such rules would be difficult of application, and of no possible utility either in numerical work or in algebra; because the required roots can be obtained more conveniently by other means, as will be explained in Chapters XIV and XV.

CHAPTER XI.

LITERAL EQUATIONS; GENERALIZATION.

292. Referring to Model I in Chapter I, we see that the problem may be stated as follows:

A man who can row a miles per hour in still water finds that it takes him b hours to row up-stream to a point from which he can return in c hours. How fast does the current flow?

As stated in Chapter I this problem had 5 instead of a, 5 instead of b, and 4 instead of c; and further on we find five other sets of figures, each of which could be substituted for a, b, and c in the statement of the problem here given.

Since we have now learned to perform all sorts of algebraic operations upon letters as well as upon figures, we can take the statement of this problem as given just above, construct our equation, and solve it, thus:

Model A.

Let $x =$ the number of miles per hour current flows;
$a + x =$ the number of miles per hour man rows downstream;
$a - x =$ the number of miles per hour man rows upstream.

① $b(a - x) = c(a + x)$	
② $ab - bx = ac + cx$	same as ①
③ $ab - ac = bx + cx$	② $+ bx - ac$
④ $ab - ac = (b + c)x$	same as ③
⑤ $x = \dfrac{ab - ac}{b + c} \equiv \dfrac{a(b - c)}{b + c}$	④ $\div (b + a)$

293. An equation, like (1), which contains letters (instead of numbers) to represent the numbers that would, ordinarily, be given in the statement of the problem, is called a **literal equation**; the answer, an algebraic expression, is in fact a **formula**, and indicates all the arithmetical operations that must be performed upon the given numbers to get the numerical answer to the problem for any particular numerical case. Thus if the problem were stated:

A man who can row $5\frac{1}{2}$ miles an hour in still water finds that it takes him $3\frac{1}{2}$ hours to row up-stream to a point from which he can return in 2 hours. How fast does the current flow?

—we should substitute for a, b, and c in the formula above, and get $x = \dfrac{5\frac{1}{2}(3\frac{1}{2} - 2)}{3\frac{1}{2} + 2} = 3\frac{1}{2} - 2 = 1\frac{1}{2}$ miles per hour.

EXERCISE CXLVI.

Similarly, by substituting in the formula, get the values of the speed of the current corresponding to the other sets of values of a, b, and c given for this problem in § 26.

Obtain a formula and substitute as before for example 1 *in Exercise XI.*

294. The degree of a term in a literal equation is the number of unknown letters that appear in that term as factors; the other letters representing such numbers as would be numerically given in the statement of the problem to be solved by this equation.

295. In literal equations of the first degree with one unknown letter, the known terms are arranged on one side of the equation, and on the other side the unknown letter, say x, is then recognized and separated as a distributive factor. The OTHER factor is the coefficient of x.

Model B.

$$bx + 3a^2x + a^3 = 3b^2x + b^3 + ax.$$

$$a^3 - b^3 = ax - bx - 3a^2x + 3b^2x$$

$$x(a - b - 3a^2 + 3b^2) = a^3 - b^3$$

$$x = \frac{a^3 - b^3}{(a-b)(1 - 3a - 3b)} \equiv \frac{a^2 + ab + b^2}{1 - 3a - 3b}$$

EXERCISE CXLVII.

1. $\dfrac{x}{a+b} + \dfrac{x}{1-a-b} = \dfrac{a+b}{1+a+b}$. 2. $(a+b)x + \dfrac{1}{a-b} = x + a - b$.

3. $\dfrac{x-b}{c} + \dfrac{x-c}{a} + \dfrac{x-a}{b} = \dfrac{x-a}{abc} - \dfrac{1}{ac} - \dfrac{1}{ab}$.

4. $ac\dfrac{p}{q} - \dfrac{(a+b)^2}{a}x - bx = ap - 36x$.

5. $(h+k)x - \dfrac{3(h+k) - kx}{2} = \dfrac{1}{2}$.

6. $(a-x)(p-x) = 3(b-x)^2$.

7. $\dfrac{x^2 - hx + k}{x - h} = \dfrac{x^2 - px + q}{x - p}$.

8. $\dfrac{(x-a)^2}{b} + \dfrac{(x-b)^2}{c} + \dfrac{(x-c)^2}{a} = (bc + ca + ab)\dfrac{x^2}{abc}$.

9. $\dfrac{3(x+a)}{k-x} + \dfrac{p(x-k)}{x+a} = p - 3$.

10. $\dfrac{x-a}{a+b} + \dfrac{x-b}{b+c} + \dfrac{x-c}{c+a} = \dfrac{x(a+b+c)^2}{(a+b)(b+c)(c+a)}$.

Model C.—To pay a sum of n dollars, p coins, some of value a cents each and the rest of value b cents each, are found to be sufficient. How many coins of each kind?

Let $x =$ the number of a-cent coins;

$p - x =$ the number of b-cent coins.

$$\text{(1)}\ ax + bp - bx = 100n$$

$$\text{(2)}\ ax - bx = 100n - bp$$

$$\text{(3)}\ x(a - b) = 100n - bp$$

$$\text{(4)}\ x = \frac{100n - bp}{a - b}\text{(no. of coins worth } a \text{ cts. apiece)}$$

$$\text{(5)}\ p - x = p - \frac{100n - bp}{a - b} \equiv \frac{ap - bp - 100n + bp}{a - b}$$

$$\equiv \frac{ap - 100n}{a - b}\text{(no. of coins worth } b \text{ cts. apiece)}$$

EXERCISE CXLVIII.

1. A father is a times as old as his son; p years ago he was b times as old. What are their ages now?

2. There is a difference of q cents a pound in the price of two commodities; a pounds of one and b pounds of the other amount to k dollars. Price of each commodity per pound?

3. Paid s dollars with coins worth a cents and coins worth b cents; d more coins of the first kind than of the second. Number of coins of each kind?

4. I have m dollars, in nickels, dimes, and 3-cent pieces; a times as many dimes as nickels, b times as many 3-cent pieces as dimes. Number of each?

5. A merchant has grain worth a cents per peck and other grain worth b cents per peck; in what proportion must he mix m bushels so that the mixture may be worth q dollars in all?

6. A man who can row s miles per hour in still water finds that he can row down-stream in a hours to a point from which it takes him b hours to return. How fast does the current flow?

7. A man who can row a miles per hour finds that it takes him r times as long to go up-stream as to go down. Find the speed of the current.

8. Bought cards a for a nickel, and the same number of other cards b for a nickel; then sold them $(a+b)$ for a dime, and lost p cents. How many cards were bought?

9. Walking, a certain distance can be accomplished in a hours; riding, in b hours; walking half the time and riding the other half, how long does it take for that distance?

10. The fore wheel of a carriage has a circumference of a feet, the hind wheel of b feet; when the fore wheel has made c turns more than the hind wheel, how far has the carriage gone?

11. A can do a certain job in a days, B in b days; how long for both?

Factoring Literal Quadratics.

296. In solving literal quadratics the equation should first be arranged in the standard form

$$ax^2 + bx + c = 0$$

and then factored by the method of cross-multiplication.*

* In discussing literal equations it is important not to confuse the letters a, b, and c which represent the three coefficients of the quadratic with similar letters which may occur in the statement of the equation itself.

Model D. Solve $x - a^2 = 2a^2 + b^2 - \dfrac{4a^2b^2}{x+a^2}$.

(2) $x^2 - a^4 = 2a^2x + b^2x + 2a^4 + a^2b^2 - 4a^2b^2$ (1) $\times (x+a^2)$

(3) $x^2 - 2a^2x - b^2x - 3a^4 + 3a^2b^2$ (2) $-2a^2x - b^2x - 2a^4 + 3a^2b^2$

(4) $x - 3a^2 = 0$ from (3)

(5) $x + (a^2 - b^2) = 0$ by Ax. A. (See Chapter V.)

(6) $x = 3a^2$; $x = b^2 - a^2$. *Ans.*

EXERCISE CXLIX.

1. $x^2 - (3a - 4)x + 2a^2 - 5a + 3 = 0.$
2. $x^2 + (2a + b)x + a^2 + ab - 2b^2 = 0.$
3. $x^2 - (a - b)x - (6a^2 + 13ab + 6b^2) = 0.$
4. $x^2 + 7x + 10 = 4ax + 17a - 3a^2.$
5. $\dfrac{x^2 - (a^2 + 13b^2)}{3a + 2b} + a = b + x.$
6. $x^2 - 60(a^2 - b^2) = x(196 - 11a).$
7. $x + a^2 = \dfrac{a^3 + a - 1}{1 + x}.$
8. $x + 1 = \dfrac{a(a^3 - x)}{a + x}.$
9. $\dfrac{x}{12a} + \dfrac{a^2 - \dfrac{x}{2}}{9} = \dfrac{x - 6a}{16}.$
10. $\dfrac{x + a^2}{4} = \sqrt{x + \dfrac{137a^2}{16}} - 4.$

297. The operation of factoring quadratics becomes more difficult when neither straight product is a monomial.

Model E.

$$(9b^2 - 4a^2)x^2 + (4ab - 4a^2)x - (a^2 - 2ab + b^2) = 0.$$

$$\begin{array}{ll} (3b + 2a)x & + (a - b) \\ (3b - 2a)x & - (a - b) \end{array}$$

$$x = \frac{b - a}{2a + 3b};\ x = \frac{b - a}{2a - 3b} \quad \textit{Ans.}$$

EXERCISE CL.

1. $(a^2 - b^2)x^2 - 2(a^2 + b^2)x + a^2 - b^2 = 0.$
2. $(a^2 + 3a + 2)x^2 + 5x + 2 = 2a^2x + 3a - a^2.$
3. $a^2(2x^2 + 3x + 1) - b^2(x^2 + 4x + 4) = ab(x^2 + 2x).$
4. $a^2(x^2 - 1) = ab(2x^2 - 7x - 1) + b^2(3x^2 + 5x - 2).$
5. $\dfrac{a^2(2x + 3)}{1 + 2x} - \dfrac{b^2(2 + 3x)}{x + 2} = \dfrac{ab(x^2 - 1)}{(x + 2)(1 + 2x)}.$
6. $\dfrac{a^2(5x + 3)}{3 + 2x} - \dfrac{b^2(3x + 5)}{2 + 3x} = \dfrac{ab(x^2 - 1)}{(3 + 2x)(2 + 3x)}.$
7. $x(6x+11)(a^2+b^2)-31ab(x^2+1)=35a^2+b^2(6-29x^2).$
8. $\dfrac{(x^2 - 1)(a^2 + ab + b^2)}{(a + b)x} = \dfrac{3ab}{a - b} - \dfrac{abx}{a + b}.$
9. $2a(1 - 10x^2) + 2 + 5a^3x^2 = (8 - 5a - 7a^2)x.$
10. $\dfrac{ab + 6}{a + b} = \dfrac{2a(x^2 + 1)}{(a + b)(1 - x^2)} - \dfrac{x^2 + a - b - 1}{x^2 - 1}.$

Solving Literal Quadratics by Formula.

298. Equations which are very difficult to factor by inspection may be solved by the quadratic formula.

Model F.

$$2a^4-5a^3-27+15ax(a^2+3)=2ax^2(a^3+a+10)+(17a^2-18)x$$

Here $a = 2a^4 + 2a^2 + 20a$

$$b = -15a^3 + 17a^2 - 45a - 18$$

$$c = -2a^4 + 5a^3 + 27$$

$$b^2 - 4ac = 16a^8 - 40a^7 + 241a^6 - 390a^5 + 1023a^4 - 990a^3 + 197a^2 - 540a + 324$$

$$\sqrt{b^2 - 4ac} = 4a^4 - 5a^3 + 27a^2 - 15a + 18$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{15a^3 - 17a^2 + 45a + 18 \pm (4a^4 - 5a^3 + 27a^2 - 15a + 18)}{4a^4 + 4a^2 + 40a}$$

$$x = \frac{2a^3 + a^2 + 3a + 9}{2a^3 - 4a^2 + 10a}; \quad \text{or} \quad x = \frac{3 - a}{a + 2}. \quad \textit{Ans.}$$

EXERCISE CLI.

1. $(a^3 - 3a^2)x^2 + 10ax - (4a^3 - 7a + 3) = 0.$
2. $a^2x^2 + (a^3 + 2a)x + a^3 + 6 = 3ax^2 + (2a^2 + 9)x + 7a.$
3. $(2a^3 + 5a^2)x^2 + (3a^3 - 2)x = 2a^3 - 5a^2 + 4 + ax - 2ax^2.$
4. $(a^3 - 1)x^2 + (4a^3 + 4)x + 3a^3 + 11a^2$
$= 4 + (a^2 - a)x^2 + (2a - 4a^2)x.$
5. $(3a^2 - 4) + (2a^2 + 3a + 2)x = (a^3 + a^2)x^2 + a^3 + 2a^3x.$
6. $a^3(x^3 + 4x + 3) - (9x + 5)x = x(x - 4a^2) + 25a + 6.$
7. $2(2x-1)(x+2) + a^2(2x+a)^2 = -2a^2x - a(4x^2 + 2x + 3).$
8. $a^3(2x^2 + 5x + 3) - (9a - 2)x^2 - 143a + 28$
$= x + (67a - 5a^2)x.$
9. $a^3(2ax^2 - 3) + (5a^2 + 7a - 2)x = 2(a^4 + ax^2) - 5 - 7a^3x.$
10. $2a^3(a + x^2) + 8x(ax - 1)$
$= 11a + 10 - x(4a^4 + 9a^3 + 14a^2 + 14a).$

299. With practice even these may be factored by inspection. In Model F, a and b may be presumed to have integral factors which can be found by trial; the factors of the straight products having been found, it should then be possible to arrange them so as to give the correct cross products. For the expert student it may safely be said that the most economical way to factor a literal quadratic is by inspection.

Literal Simultaneous Equations.

300. Model G.

$$
\begin{array}{lll}
① & a_1x + b_1y = c_1 & \\
② & a_2x + b_2y = c_2 & \\
③ & a_2a_1x + a_2b_1y = a_2c_1 & ① \times a_2 \\
④ & a_1a_2x + a_1b_2y = a_1c_2 & ② \times a_1 \\
⑤ & (a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1 & ④ - ③ \\
⑥ & y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} & ⑤ \div (a_1b_2 - a_2b_1)
\end{array}
$$

Similarly we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

It is generally cheaper to eliminate for each letter independently than to substitute a complicated literal expression for one, to find the other.

EXERCISE CLII.

1. $c_1x + s_1y = p_1.$
   $c_2x + s_2y = p_2.$

2. $x + a_1y = 1.$
   $a_2x + y = 1.$

3. $y = m_1x + b_1.$
   $y = m_2x + b_2.$

4. $\dfrac{x}{a} + \dfrac{y}{b} = 1.$
   $x + y = a.$

5. $x + y = a.$
   $x - y = b.$

6. $x + ay = b.$
   $bx + y = a.$

7. $ax + b = y.$
   $x + y = a.$

8. $x + y = a_1.$
   $bx + y = a_2.$

9. $\dfrac{x}{a_1} + \dfrac{y}{b_1} = 1.$
   $\dfrac{x}{a_2} + \dfrac{y}{b_2} = 1.$

10. $a(b + x^2) = x^2\left(a + \dfrac{b}{x}\right).$
    $ax = by.$

11. $\left.\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}\right\}$ Find x only.

12. $ax + by + z = c.$
$x + ay + bz = c.$
$bx + y + az = c.$

13. $x + ay = b.$
$y + bz = c.$
$z + cx = a.$

14. $x - ay = b.$
$y - bz = c.$
$z - cx = a.$

15. $x + y + z = a_1.$
$x - y + z = a_2.$
$x - y - z = a_3.$

16. $2z + x - y = a_2 + \frac{y}{2}.$
$y - x + 2z = a_1 - \frac{y}{2}.$
$x - 2z + y = a_3 - \frac{y}{2}.$

17. $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3.$
$\frac{a}{x} - \frac{b}{y} = 5.$
$\frac{c}{2z} = \frac{a}{x}.$

18. $x + y + z = 0.$
$x - y = a_1 - 2a_2 + a_3.$
$x + 2y + 3z = a_3 - 2a_1 - a_2.$

19. $x + 2y = a + 4b + 3.$
$y + 2z = 2b + 6c + 3.$
$z + 2x = 2a + 3c + 3.$

20. $x + y + z + w = 2a + 2b.$
$x - 2y + z = b - a = w + 3y - 2a.$
$x + y - z - w = 2a - 2b - 2c.$

21. $2x + 3y = a; \quad \frac{2x(3y + x)}{2a + 3y} = \frac{a}{4}.$

22. $\frac{a + b}{x} = \frac{x}{4y + x}; \quad 2x + 4y = a.$

23. $x + a - b = y; \quad \frac{x^2 - ab}{y - a} = \frac{y + b}{2}.$

24. $x - 3a = 2y; \quad \frac{a + b}{2y + a} + \frac{2a + b}{a} = \frac{2y + 3a}{a}.$

25. $y = a + b + x; \quad \frac{y - 2b}{y} + \frac{y - x}{y - a} = 2.$

26. $y = x + a - 4b;\quad \dfrac{a+4b}{x+2b} - \dfrac{y-x}{x-2b} = \dfrac{x+a-y}{a}.$

27. $\dfrac{x}{a} + \dfrac{y}{b} = 1;\quad \dfrac{a}{x} + \dfrac{y}{b} = 4.$

28. $x + y = 5a;\quad xy : b = 5a - b.$

29. $x^2 + axy + ay^2 = xy;\quad x + y = 3a.$

30. $x^2 + y^2 = a^2xy;\quad x + y = bxy.$

Constants and Variables.

301. In the Chapter on Elimination we saw that in the equation

$$2x - 3y = 6,$$

while x and y are restricted, still the value of x may VARY from any negative number to any positive number; and the value of y changes with every value of x.

Similarly in the equation $ax - by = c$ the values of x and y could vary, subject only to the restriction that, whatever the value of x, y must be given by the formula $y = \dfrac{ax - c}{b}$; and a, b, and c, like the 2, 3, and 6 in the numerical equation, remain unchanged throughout.

302. In any problem the numbers whose values are given are called **constants** of the problem; and those whose values are required are called **variables**.

Thus in Model A of this chapter the numbers a, b, and c are the CONSTANTS; the (unknown) speed of the current is the VARIABLE.

As the speed of the current is supposed to have a certain fixed value, which we are trying to find when we start to solve the problem, it seems strange to call that speed a VARIABLE; but we may state for this problem three

equations of condition, in any one of which, taken separately, x, or y, or z, is a variable, subject to a restriction.

If we let $y =$ the speed of the boat down-stream, and $z =$ the speed of the boat up-stream; then $y = a + x$; $z = a - x$; $bz = cy$. So far as the first of these equations is concerned, x may have any value, so long as y is greater than it by a. But no algebraic restatement can make a, or b, or c appear to be VARIABLE; they are the CONSTANTS of the problem.

Discussion of Problems.

303. A problem is said to be **generalized** when one or more letters are used instead of numerical constants.

304. The investigation of a generalized answer, with a view to determining the effect upon it produced by different values or relations imposed upon the constants, is called **Discussion of the problem.**

LIMITING VALUES.

305. In discussing a generalized problem we often meet values which by the ordinary rules of algebra have no meaning whatever. The two most important of such results are, first, that obtained when the denominator of a formula becomes 0 for a particular set of values for the constants; and second, that obtained when the numerator and denominator both become 0. It is usual to consider the following three cases together:

(I) Where the numerator is 0;

(II) Where the denominator is 0;

(III) Where both numerator and denominator are 0.

These are called **limiting values** of the formula under discussion.

Model H.—If we set

$$y = \frac{x^2 - 8x + 15}{x^2 - 7x + 12}$$

we obtain the following limiting values:

$$\text{(I)} \qquad x = 5;\ y = \frac{25 - 40 + 15}{25 - 35 + 12} = \frac{0}{2}$$

$$\text{(II)} \qquad x = 4;\ y = \frac{16 - 32 + 15}{16 - 28 + 12} = -\frac{1}{0}$$

$$\text{(III)} \qquad x = 3;\ y = \frac{9 - 24 + 15}{9 - 21 + 12} = \frac{0}{0}$$

In considering (I) we may conclude at once that since if 0 is divided into 2 parts, the result will be zero; and since if 2 is divided into zero,* the quotient and the remainder will both be zero; then the value of $\frac{0}{2}$ must be 0.

When we come to (II) we have no such easy task, since to divide 1 by 0 is meaningless; and still more meaningless, in (III), is $\frac{0}{0}$.

We have to investigate then the three expressions (I) $\frac{0}{a}$; (II) $\frac{b}{0}$; (III) $\frac{0}{0}$.

Let us consider at first the three expressions (I) $\frac{x}{a}$; (II) $\frac{b}{y}$; (III) $\frac{x}{y}$; where x and y are variables, and a and b are positive constants.

(I) So long as x is positive, $\frac{x}{a}$ will be positive; as x decreases, $\frac{x}{a}$ decreases also; when x passes over from a

* As, for example, when one is asked how many times he can fill a two-quart measure from an empty bin.

small $+$ value to a small $-$ value, $\frac{x}{a}$ does likewise. At the instant when x, in its progress from $+$ values to $-$ values, assumes the value $x = 0$, $\frac{x}{a}$ at the same instant assumes the value 0.

We may conclude, then, that

$$\text{(I)}\quad \frac{\mathbf{0}}{a} = \mathbf{0}$$

and also we may say that

(I) **$\frac{\mathbf{0}}{a}$ is the limit which $\frac{x}{a}$ approaches as the numerical value of x is indefinitely diminished.**

If a had been a negative constant, then $\frac{x}{a}$ and x would have opposite signs just so long as x was $+$; but at the instant when x passed from $+$ to $-$, $\frac{x}{a}$ would pass from $-$ to $+$, and, as before, we should have $\frac{0}{a} = 0$.

(II) If now we suppose y to begin with a considerable $+$ value and then to diminish, $\frac{b}{y}$ would not diminish, but increase, on the principle that, the dividend remaining the same, a smaller divisor gives a larger quotient. If y becomes $-$, $\frac{b}{y}$ also becomes $-$.

As y passes through zero, from the region of $+$ numbers to the region of $-$ numbers, $\frac{b}{y}$ passes also from the region of $+$ numbers to the region of $-$ numbers; but it does NOT pass through 0. We conclude then that

(II) **$\frac{b}{\mathbf{0}}$, like 0, is a boundary between $+$ and $-$ numbers.**

Again, by causing y to assume a very small value, positive or negative, and to become successively smaller and smaller, we cause $\frac{b}{y}$ to assume a very LARGE value positive or negative, and to become successively larger and larger. By choosing y suitably we can cause $\frac{b}{y}$ to assume a value greater than any assignable value, however large; and still, if y is then taken NEARER to 0 in value, $\frac{b}{y}$ becomes even larger. Hence we also conclude:

(II) $\frac{b}{0}$ **is a limit towards which large positive or large negative numbers may approach if we suppose them to increase indefinitely.**

The symbol for this limit is ∞ and its name is **infinity.**

$$\text{(II)}\ \frac{b}{0} = \infty.$$

(III) If we suppose the expression $\frac{x}{y}$ equal to any constant k, we could still allow x to vary, and to assume any value whatever, positive or negative; y would always be determined, for any particular value of x, by the formula $y = \frac{x}{k}$. We may cause both x and y to assume values less than any assignable quantity, however small, without destroying the equation $\frac{x}{y} = k$. In other words, we may cause $\frac{x}{y}$ to approach as near as we please to $\frac{0}{0}$ without destroying the equation $\frac{x}{y} = k$. These considerations will all hold good whatever the value of k; hence we may

conclude that the symbol $\frac{0}{0}$ does not of itself determine any value.

(III) $\mathbf{\frac{0}{0}}$ **is indeterminate.**

This conclusion does not mean that $\frac{0}{0}$ = ANY number; only that when substitution in a formula leads to this result, some special reduction must be resorted to for determining its actual value, if the value can be determined at all.

Model H.—In the expression

$$y = \frac{x^2 - 8x + 15}{x^2 - 7x + 12}$$

if $x = 3$, $y = \frac{9 - 24 + 15}{9 - 21 + 12} = \frac{0}{0}$;

but $$y = \frac{x^2 - 8x + 15}{x^2 - 7x + 12} \equiv \frac{(x-3)(x-5)}{(x-3)(x-4)} \equiv \frac{x-5}{x-4} = 2.$$

Problems for Discussion.

306. Model I.—A cistern can be filled by one pipe in 12 minutes, and emptied by another in 36 minutes. How long will it take to fill it with both pipes open?

Generalized statement:

A cistern can be filled by one pipe in a minutes, and emptied by another in b minutes. How long will it take to fill it with both pipes open?

x = number of minutes to fill, both pipes open.

$\frac{1}{a}$ = fraction of cistern filled by first pipe in one minute.

$\frac{1}{b}$ = fraction of cistern emptied by second pipe in one minute.

$$① \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{x}$$

$$② \quad bx - ax = ab \qquad ① \times abx$$

$$③ \quad x = \frac{ab}{b - a} \qquad ② \div (b - a)$$

Now if b is greater than a, this answer is positive, and its meaning is natural and obvious. Thus in the statement of the problem first given, $a = 12$, $b = 36$; then

$$x = \frac{12 \times 36}{36 - 12} = 18 \text{ minutes to fill.}$$

If b is less than a, the answer is negative; on reflection, we see that if the outlet emptied in less time than the supply-pipe filled (the emptying would then be more rapid than the filling), it would be vain to expect the cistern to fill with both open. Assuming $a = 24$, $b = 16$, we obtain

$$x = \frac{24 \times 16}{16 - 24} = -48.$$

We might say that while in the first case the instant of being full is 18 minutes AFTER the instant of being empty, in the second case the instant of being full is 48 minutes BEFORE the instant of being empty.

If b were negative, the numerator, ba, product of unlike factors, would be $-$; the denominator, $b - a$, sum of two negative terms, would also be $-$; so x would be plus.

Now for b to be a negative number is the same as to say FILL instead of empty; and with both pipes filling, the answer would evidently be $+$ and reasonable. Thus, if $a = 28$, $b = -21$,

$$x = \frac{28 \times (-21)}{-21 - 28} = +\frac{28 \times 21}{49} = 12.$$

If a is negative, the numerator ab will be negative and the denominator $b - a$ will be positive; so x will be $-$.

But for a to be $-$ is to have the supply-pipe also an outlet; and in that case (as before when $b < a$) x is $-$ and the cistern is being emptied instead of filled.

If $a = b$, x assumes a limiting value; $b - a = 0$ and

$$x = \frac{ab}{b - a} = \frac{a^2}{0} = \infty .$$

In this case one pipe empties as fast as the other pipe fills, and the cistern will never be any fuller than it is now.

EXERCISE CLIII.

In a similar way generalize and discuss the following problems:

1. A boat travelling 10 miles per hour passed south by Highland light at noon; a steamer pursuing at 13 miles per hour passed on the same track 3 hours later. When will the steamer overtake the boat?

2. Of two pipes running into a cistern each could fill it in 21 minutes; the waste-pipe could empty it in 35 minutes. How long would it take to fill the cistern with all pipes open?

3. A person who can row 5 miles an hour in still water rows 21 miles down-stream and back in 10 hours, the stream flowing uniformly all the time. How many miles an hour does the stream flow?

4. How much rye at 50 cents a bushel must be mixed with 50 bushels of wheat at 80 cents a bushel to make a mixture worth 70 cents a bushel?

5. A bicyclist starts a ride, going 22 feet per second; half a minute later a second rider starts 60 yards behind the first man's mark, riding 25 feet per second. How

far from the first man's mark will the riders be side by side?

Discuss the solutions of the examples in Exercise CXLVIII.

DISCUSSION OF EQUATIONS.

307. In the pair of linear equations

$$\text{①}\ ax + by = c$$
$$\text{②}\ px + qy = r$$

we obtain by elimination

$$x = \frac{cq - br}{aq - bp}; \quad y = \frac{ar - pc}{aq - bp}.$$

Of these two answers, $x = 0$ if $cq - br = 0$ and $aq - bp \neq 0$*; $y = 0$ if $ar - pc = 0$ and $aq - bp \neq 0$. If $aq - bp = 0$ and $cq - br \neq 0$, $x = \infty$; and if $aq - bp = 0$ and $cq - br = 0$, $x = \frac{0}{0}$, that is, its value is undetermined by these equations.

308. In considering the special cases just described, what we have to decide is whether

$$(1) \quad aq - bp = 0$$
$$(2) \quad cq - br = 0$$
$$(3) \quad ar - pc = 0$$

From these three conditions we get

$$\left.\begin{array}{l} aq = bp \\ \dfrac{a}{p} = \dfrac{b}{q} \end{array}\right\} \text{from (1)}$$

and similarly $\quad \dfrac{c}{r} = \dfrac{b}{q} \quad$ from (2)

and $\quad \dfrac{a}{p} = \dfrac{c}{r} \quad$ from (3)

* The sign $\neq$ means "is not equal to."

It is evident that if any two of these conditions are satisfied, the third must be.

309. Let us now suppose that only one of these conditions is satisfied. Suppose

$$(1) \quad aq - bp = 0; \quad \text{i.e. } \frac{a}{p} = \frac{b}{q}$$

Two equations that satisfy this condition are

Model J.

$$\begin{aligned} ① \; 3x + 7y &= 5 \\ ② \; 6x + 14y &= 6 \end{aligned}$$

Here $\qquad \frac{a}{p} = \frac{3}{6} = \frac{1}{2}; \quad \frac{b}{q} = \frac{7}{14} = \frac{1}{2}$

$$x = \frac{70 - 42}{42 - 42} = \frac{28}{0}; \quad y = \frac{18 - 30}{0} = -\frac{12}{0}$$

that is, $\qquad x = \infty; \quad y = \infty$

If we drew a diagram representing the infinite lists of answers to these equations, as in Chapter VI, we should find them to be a pair of PARALLEL straight lines, which do not intersect, or, in other words, which intersect at an infinite distance.

These two equations are, in fact, INCONSISTENT; for from the first we have ① $3x + 7y = 5$; and from the second $3x + 7y = 3$ ② $\div 2$; and $3x + 7y$ cannot at the same time be equal to 3 and also to 5.

310. If two (and therefore all) of the three conditions are satisfied, we should have a pair of equations that would NOT be INDEPENDENT. For in that case $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$; that is, one equation would be the same as the other except for being multiplied through by some constant.

Model K.

$$\begin{aligned} ① \; 2x + 3y &= 5 \\ ② \; 10x + 15y &= 25 \end{aligned}$$

$$x = \frac{75 - 75}{30 - 30}; \quad y = \frac{50 - 50}{30 - 30}; \quad x = y = \frac{0}{0}.$$

The diagram for each of these equations would be the same straight line.

311. The discussion of a generalized set of equations with more than two unknown quantities is very much more complicated, and cannot be treated here.

Discussion of the Quadratic.

312. The generalized quadratic equation

$$ax^2 + bx + c = 0$$

whose solution is familiar

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

presents important features for discussion.

Two values of x are given by the formula:

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac$ is $+$, α and β are both **real,*** **and different.**
Thus in $5x^2 + 7x - 1 = 0$

$$x = \frac{-7 \pm \sqrt{79}}{10}.$$

If $b^2 - 4ac = 0$, and β are **real, and alike.**
Thus in $9x^2 + 30x + 25 = 0$

$$x = \frac{-15 \pm \sqrt{0}}{18} = -\frac{15}{18}.$$

* I.e., not imaginary.

If $b^2 - 4ac$ is $-$, $\sqrt{b^2 - 4ac}$ will be imaginary, and the values of α and β will be **conjugate imaginaries.***

Thus in $2x^2 + 3x + 5 = 0$

$$x = \frac{-3 \pm \sqrt{-31}}{4}.$$

EXERCISE CLIV.

State the character of the roots (whether real or imaginary, whether equal or different) in the following equations, without solving them; and in each case give reasons:

1. $100x^2 + 8x + .001 = 0.$
2. $x^2 + 2x + 3 = 0.$
3. $x^2 + 2x - 3 = 0.$
4. † $px^2 + qx - r = 0.$
5. $7x^2 + 3x + \frac{9}{28} = 0.$
6. † $x^2 + 4p^2 = 2px.$
7. † $p = x - x^2.$
8. † $x^2 + p = 0.$
9. $1 - \sqrt{3x^2 - 2x + 3} = 0.$
10. $\frac{15 - x}{x + 4} - 10x = 15.$

Zero Coefficients in the Quadratic.

313. If c becomes 0, while a and b do not,

$$\alpha = \frac{-b + \sqrt{b^2 - 0}}{2a} \equiv \frac{0}{2a} \equiv 0$$

$$\beta = \frac{-b - \sqrt{b^2 - 0}}{2a} \equiv -\frac{b}{a}$$

which are obviously the answers to $ax^2 + bx = 0$.

* See § 274.

† In this example suppose p, q, and r positive integers.

314. If b becomes 0, while a and c do not,

$$\alpha = \frac{0 + \sqrt{-4ac}}{2a} \quad \text{and} \quad \beta = \frac{-\sqrt{-4ac}}{2a}$$

which reduce to

$$\alpha = \sqrt{-\frac{c}{a}}; \quad \beta = -\sqrt{-\frac{c}{a}}.$$

Such would obviously be the answers to

$$ax^2 + c = 0.$$

315. If b and c both become 0,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{0 + \sqrt{0}}{2a} \equiv \frac{0}{2a} \equiv 0$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{0}{2a} \equiv 0$$

which are obviously the answers to $ax^2 = 0$.

316. If a becomes 0, the equation reduces at once to an equation of the first degree, $bx + c = 0$, and $x = -\frac{c}{b}$. But it is interesting to note what becomes of the TWO answers, α and β, as the coefficient a assumes values that are very small as compared with b and c. When a becomes VERY small indeed as compared with b and c, the quadratic becomes VERY NEARLY the same as the linear equation $bx + c = 0$, but still has its two answers, α and β.

Now in $ax^2 + bx + c = 0$ if $a = 0$,

$$\alpha = \frac{-b + \sqrt{b^2}}{0} \equiv \frac{0}{0}$$

$$\beta = \frac{-b - \sqrt{b^2}}{0} \equiv \frac{-2b}{0} \equiv \infty$$

The value of α cannot be determined from the formula, in this case; but if we rationalize the numerator of the formula by multiplying it by its conjugate surd,* we get

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}\cdot\frac{(-b-\sqrt{b^2-4ac})}{(-b-\sqrt{b^2-4ac})} \equiv \frac{b^2-(b^2-4ac)}{2a(-b-\sqrt{b^2-4ac})}$$

$$\equiv \frac{4ac}{2a(-b-\sqrt{b^2-4ac})} \equiv \frac{2c}{-b-\sqrt{b^2-4ac}}.$$

If now in this formula we let $a = 0$, we get

$$\alpha = \frac{2c}{-b-b} \equiv -\frac{c}{b}.$$

The meaning of these conclusions is this: in any quadratic if the coefficient of x^2 becomes very small, as compared with the other two coefficients, one answer becomes very large, and the other answer becomes nearly equal to the value x would have if the first term were removed altogether.

Model I.—In the equation

$$\frac{x}{10^n} + 2(10)^n x + (10)^n = 0$$

if $n = 1$, we have $\alpha = -.502$ and $\beta = -200$ (nearly); if $n = 2$, $\alpha = .5$ (very nearly) and $\beta = -20000$ (nearly); and the larger we take n the more nearly α becomes $= .5$ and the larger β becomes.

317. If a and b both become 0, while c does not,

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{0}{0},$$

but by the other formula $\alpha = \dfrac{-2c}{b+\sqrt{b^2-4ac}} = \dfrac{-2c}{0} \equiv \infty$;

* See § 274.

and by a second formula for β, similarly obtained,

$$\beta = \frac{-2c}{b - \sqrt{b^2 - 4ac}} = \frac{-2c}{0} \equiv \infty.$$

In this case, if a and b were really zero, we should have c (a constant) equal to zero; that would be, of course, a meaningless statement. Thus $0.x^2 + 0.x + 5 = 0$, being the same as $5 = 0$, could not be true. But if a and b both became VERY SMALL AS COMPARED WITH c, it would take larger and larger values of x to satisfy the equation.

318. If a and c both become 0, while b does not,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{b}}{0} \equiv \frac{0}{0}$$

by the other formula, $\alpha = \dfrac{-2c}{b + \sqrt{b^2 - 4ac}} = \dfrac{0}{2b} \equiv 0$

while $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{-2b}{0} \equiv \infty.$

If a and c both become VERY SMALL as compared with b, then α will become VERY SMALL and β will become VERY LARGE.

EXERCISE CLV.

Determine the limiting values of x as n increases indefinitely:

1. $10^n x^2 + \dfrac{2x}{10^n} + 10^n = 0.$

2. $3^n x^2 + 2(3)^n x + \dfrac{1}{3^n} = 0.$

3. $2^n x^2 - 2^{n+1} x + \dfrac{1}{2^n} = 0.$

4. $10^n x^2 + \dfrac{x}{10^n} + \dfrac{1}{10^n} = 0.$

5. $\dfrac{x^2}{10^n} - 10^n x + \dfrac{1}{10^n} = 0.$

6. $\dfrac{x^2}{10^n} - \dfrac{x}{10^n} + 10^n = 0.$

CHAPTER XII.

PROPORTION AND VARIATION.

319. An equation between two ratios is called a PROPORTION; the theory of such equations can be investigated by means of a generalized type, and the laws so obtained expressed as theorems.

The general type of a proportion may be represented by

$$① \; \frac{x}{y} = \frac{a}{b}$$

Of these four quantities $x, y, a, b,$ x and b are called the **extremes**, y and a the **means**; x and a are the **antecedents**, y and b the **consequents**; x and y form the **first ratio**, a and b the **second ratio**.

The Three Simplest Theorems.

320. If we assume ① to be true, equations ②, ③, and ④ are at once obtained, and the theorems they express are consequently true of any proportion.

② $bx = ay$ ① $\times\, by$

I. The product of the means is equal to the product of the extremes.

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③ $\dfrac{x}{a} = \dfrac{y}{b}$ ① $\times\, \dfrac{y}{a}$

II. The ratio of the antecedents is equal to the ratio of the consequents.

④ $\dfrac{y}{x}=\dfrac{b}{a}$ $\quad 1 \div$ ① III. **The reciprocals of the ratios are equal.**

---

321. Of these three theorems the first enables us to find any term of the proportion when the other three are given, —a method known to our grandfathers as the "**Rule of Three.**" The second theorem is known as the "**Law of Alternation,**" and the third as the "**Law of Inversion.**" These three are the most elementary transformations of a proportion.

EXERCISE CLVI.

1. Find x in the proportion $\dfrac{2}{9}=\dfrac{7}{x}$.

2. Find x in the proportion $(x-10):2=\dfrac{7}{x+3}$.

3. Find x in the proportion $\dfrac{2x+3}{6}=\dfrac{x+2}{3x-1}$.

4. By what successive transformations may we obtain from $x:y=a:b$ the following forms?

$$x:a=y:b$$
$$b:y=a:x$$
$$y:b=x:a$$

5. Prove that if $x:y=a:b$, then $x:my=a:mb$.

The Other Theorems of Proportion.

322. Returning to the generalized proportion

$$① \quad \frac{x}{y}=\frac{a}{b}$$

we may represent the value of each of these two equal ratios by some letter, say r.

$$① \quad \frac{x}{y} = \frac{a}{b} = r$$

Hence $x = ry$ and $a = rb$.

Then $$⑤ \quad \frac{x+y}{y} = \frac{ry+y}{y} \equiv \frac{r+1}{1} = \frac{a+b}{b}$$

and $$⑥ \quad \frac{x+y}{x} = \frac{r+1}{r} = \frac{a+b}{b}$$

IV. **In any proportion equal ratios are obtained by dividing the sum of the terms of each ratio by its antecedent or by its consequent.** This is called the **Law of Composition.**

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323. Again $$⑦ \quad \frac{x-y}{y} = \frac{ry-y}{y} \equiv \frac{r-1}{1} = \frac{a-b}{b}$$

and $$⑧ \quad \frac{x-y}{x} = \frac{r-1}{r} = \frac{a-b}{a}$$

V. **In any proportion equal ratios are obtained by dividing the difference of the terms of each ratio by its antecedent or by its consequent.** This is called the **Law of Division.**

---

324. Again $$⑨ \quad \frac{x+y}{x-y} = \frac{ry+y}{ry-y} \equiv \frac{r+1}{r-1} = \frac{a+b}{a-b}$$

that is $$\frac{x+y}{x-y} = \frac{a+b}{a-b}$$

VI. **In any proportion equal ratios are obtained by dividing the sum of the terms of each ratio by their difference.** This is called the **Law of Composition and Division.**

325. Of the four terms of any proportion, two are said to be **homologous** if they are both antecedents, both consequents, or both terms of the same ratio.

326. Homologous terms, if not terms of the same ratio, may always be made such by Alternation; and since the value of a ratio is not altered by multiplying antecedent and consequent * by the same number, we have this transformation:

VII. **In any proportion, if any pair of homologous terms be multiplied each by the same number, the resulting proportion is true.**

---

327. Returning to our first proportion,—

$$\frac{x^n}{y^n} = \frac{a^n}{b^n} \quad (1)^n$$

VIII. **In any proportion, if like powers be taken of all four terms, the four results will be in proportion.**

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328. If we have two proportions

$$x_1 : y_1 = a_1 : b_1; \text{ and } x_2 : y_2 = a_2 : b_2; \text{ then}$$

IX. **The products of corresponding terms will be in proportion**; that is, multiplying the two equations,

$$x_1x_2 : y_1y_2 = a_1a_2 : b_1b_2.$$

---

329. As in VI we may prove that if

$$\frac{x}{y} = \frac{a}{b} = \frac{p}{q} = \frac{h}{k} = \frac{d}{s} = \ldots$$

then $$\frac{x + a + p + h + d + \ldots}{y + b + q + k + s + \ldots} = \frac{x}{y}; \text{ that is:}$$

* Numerator and denominator.

X. In any SERIES of equal ratios, the sum of all the antecedents divided by the sum of all the consequents is equal to the ratio of any antecedent to its consequent.

The equation $\frac{x}{y}=\frac{a}{b}$ is sometimes read "As x is to y, so is a to b"; or, "x is to y, as a to b"; and this sentence is the expression of the statement that the four quantities, x, y, a, and b, are in proportion; or that x and y are proportional to a and b. Instead of the equal sign, the sign :: was generally used; and the proportion was sometimes called an "analogy." Dropping this cumbersome phraseology, much of the difficulty of the subject of proportion goes with it, and the different theorems are seen to be so many short cuts in the handling of a very simple type of fractional equations.

A very complicated and difficult definition, known as Euclid's definition of proportion, is gradually passing out of use.

EXERCISE CLVII.

Assuming that $x : y = a : b$, establish the following equations:

1. $mx : mxy = a : bx$.
2. $xy : ab = y^2 : b^2$.
3. $kmx : myz = ak : bz$.
4. $px^2+qy^2 : y^2=pa^2+qb^2 : b^2$.
5. $mx + y : ma + b = px + y : pa + b$.
6. $px + qy : px - qy = pa + qb : pa - qb$.
7. $x^2 + 2y^2 : a^2 + 2b^2 = (x + y)^2 : (a + b)^2$.
8. $2x^2 + 3y^2 : 2a^2 + 3b^2 = xy : ab$.
9. $2x^2 - 3y^2 : 2a^2 - 3b^2 = 2x^2 - 7y^2 : 2a^2 - 7b^2$.
10. $2x^2 - 3xy - 2y^2 : 2a^2 - 3ab - 2b^2 = px^2+qy^2 : pa^2+qb^2$.

Assuming that $a : x = b : y = c : z$, prove:

11. $a + 3b + 5c : a - 2c = x + 3y + 5z : x - 2z$.
12. $px + qy + z : px + qz = pa + qb + c : pa + qc$.
13. $x^3+y^3-xyz : xy^2+y^2z+z^2x=a^3+b^3-abc : ab^2+b^2c+c^2a$.
14. $\dfrac{a^2 + b^2 - bc}{x^2 + y^2 - yz} = \dfrac{ac - ba}{xz - xy}$.

15. $\dfrac{x + 2y + z}{a + 2b + c} = \dfrac{\sqrt{x^2 + y^2 + z^2 - 3xy}}{\sqrt{a^2 + b^2 + c^2 - 3ab}}$.

Use the theorems to reduce the following equations:

16. Apply to the proportion

$$\frac{2a + 5b + c + 3d}{2a - 5b + c - 3d} = \frac{2a + 5b - c - 3d}{2a - 5b - c + 3d}$$

the theorems VI, II, VI successively.

17. If $x + 2 : 7 = y + 3 : 9$, find the ratio of $x - 5$ to $y - 6$.

18. Apply the law of Composition and Division to the equation $\dfrac{4x^2 + 2x + 1}{4x^2 - 2x - 1} = \dfrac{2x^2 - x + 1}{2x^2 + x - 1}$. Then solve for x.

19. Given $x^4 + 1 : 2x^2 = 97 : 72$; find x.

20. Solve $2x^2 + 10x + 53 : 18x + 45 = 101 : 99$. (Apply VI.)

330. A proportion may be formed out of three quantities if the first divided by the second gives the same ratio as the second divided by the third. In that case the second quantity is called a **mean proportional** (or **geometric mean**) between the other two; and the third quantity is called a **third proportional** to the first two.

In any proportion the second consequent is called a **fourth proportional** to the other three quantities.

For a mean proportional the order of the extremes is immaterial; for a third or fourth proportional the other terms must be given AS THEY ARE ARRANGED IN THE PROPORTION.

331. When terms in a series are so chosen that the ratio of any consecutive pair is the same as the ratio of any

other, the quantities are said to be in **continued proportion** (or in **geometric progression**).

Thus 16; 8; 4; 2; 1; $\frac{1}{2}$; $\frac{1}{4}$; etc., are in continued proportion; so are $a, b, c, d, e, f, \ldots$ if

$$a : b = b : c = c : d = d : e = e : f = \ldots$$

Of a series of terms in continued proportion any term is a mean proportional of the two terms adjacent to it. Thus in the two examples cited, 4 is a mean proportional between 2 and 8; c is a mean proportional between b and d. (See further in the chapter on progressions.)

EXERCISE CLVIII.

1. Find a fourth proportional to 2, 3, and 4.
2. Find a fourth proportional to 3, 4, and 2.
3. Find a mean proportional to $2\frac{1}{3}$ and $9\frac{1}{3}$.
4. Find a third proportional to $2\frac{1}{3}$ and $9\frac{1}{3}$.
5. 18, x, and 50 are in continued proportion; find a fourth proportional to them, eliminating x.
6. Find a third proportional to a^3 and $3a^2b$; to $9a^2b^2$ and a^3; and of these two results find the geometric mean.
7. Of several numbers in continued proportion, if the second and third are 4 and 6, what is the fifth?
8. Find a number to which if 2, 12, and 17 be separately added, the three results are in continued proportion.
9. Of two numbers one is 7 greater and the other 3 less than their mean proportional. What are the numbers?
10. Of four numbers in continued proportion, 8 is the least and 27 the greatest. Find the other two.

VARIATION.

332. In any algebraic investigation there may be two kinds of quantities: **variables**, whose values are subject to

change, either continuously or at intervals; and **constants**, whose values are understood to be fixed and unchangeable throughout the investigation.

Thus in the expression

$$\frac{x^2 - 8x + 15}{x^2 - 7x + 12}$$

x may assume any value we choose to impose upon it; while 8, 15, 7, and 12 are constants.

333. The relation between the variables x and y in the equation

$$x = ky \quad (\textit{when the constant } k \textit{ is unknown})$$

is sometimes expressed $x \propto y$; to be read "x varies as y."

A value of x and a value of y which when substituted together in this equation make it a true statement are called **simultaneous** values, or **corresponding** values, of x and y. They may be represented with equal suffixes, thus: x_1 and y_1, x_2 and y_2, x_3 and y_3 are simultaneous pairs, and we have the equations

$$x_1 = ky_1; \qquad \frac{x_1}{y_1} = k$$

$$x_2 = ky_2; \qquad \frac{x_2}{y_2} = k$$

Hence $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$; or by alternation $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2}$.

334. If one quantity varies as another, a proportion may be formed by using, for homologous terms of the proportion, corresponding values of the variables.

Model A.—The weight of wire of a certain kind varies as its length; if 1 metre of it weighs .8 grams, how much will 2.35 metres weigh?

Here $l_1 = 1$; $w_1 = .8$; $l_2 = 2.35$.

$$1 : .8 = 2.35 : w_2$$

$$w_2 = .8 \times 2.35 = 1.88 \text{ grams.} \quad \textit{Ans.}$$

EXERCISE CLIX.

1. If x varies as y, and $x = 3$ when $y = 10$, find the value of x when $y = 11$.

2. If $x \propto a$, and $x = 1.09$ when $y = .7$, find y when $x = 23$.

3. If $x \propto z$, and $z = p$ when $x = q$, find x when $z = q$.

4. If $a \propto b$, and $a = q^2 - p^2$ when $b = q^2$, find b when $a = pq(p - q)$.

5. If $h \propto k$, and $k = \sqrt{3}$ when $h = \sqrt{5}$, find h when $k = \sqrt{6}$.

335. One variable may vary as the reciprocal of the other, or as its square or cube, or as any algebraic expression involving the other variable.

The proportion will then be constructed by taking for homologous terms the same algebraic expression with the different values of the variable. Thus if $y \propto x^2 + 3x - 5$

$$y_1 : x_1^2 + 3x_1 - 5 = y_2 : x_2^2 + 3x_2 - 5.$$

Or, if $y^2 + 2y^3 \propto x^3 - x$

$$y_1^2 + 2y_1^3 : x_1^3 - x_1 = y_2^2 + 2y_2^3 : x_2^3 - x_2.$$

336. When one variable varies as the reciprocal of another they are said to vary **inversely**; in distinction from this, the case of variation first described is called direct variation.

Thus when x varies inversely as y^2, $x \propto \frac{1}{y^2}$; when x varies directly as y^3, $x \propto y^3$.

Often the expression "varies inversely as" is replaced by the words "is reciprocally proportional to"; and "varies directly as" may be written "is directly proportional to," or simply "is proportional to."

EXERCISE CLX.

1. If an animal's strength is proportional to the square of his length, and an animal 4 feet long can pull 1250 pounds, how much can an animal 4.3 feet long pull?

2. If the weight of a body is reciprocally proportional to the square of its distance from the centre of the earth, how much weight will a 3-pound ball indicate on a spring balance, at a distance above the earth equal to the diameter of the earth?

3. If x varies inversely as $y^2 + y$, and $x = 5$ when $y = 2$, what will x equal when $y = 3$?

4. The area of an equilateral triangle varies as the square of its side; and an equilateral triangle whose side is 5 feet long incloses about 10.825 square feet. Find the area of an equilateral triangle whose side is 8 feet long.

5. If the intensity of light varies as the square of its distance, and if a certain lamp at a distance of 10 feet gives an intensity $\frac{1}{a}$ that of sunlight, what will be its intensity at a distance of 50 feet?

337. If there are three quantities so related that the first varies as the second when the third is constant, and varies as the third when the second is constant, then the first varies as the product of the second and third when they both vary.

Proof. Let $x_1y_1z_1$, $x_2y_2z_2$ be corresponding values of the three variables. If we suppose y_1 to be constant, and z to

vary from z_1 to z_2, we shall have a new value of x, which we may call x', for which this equation holds:

$$① \ \frac{x_1}{x'} = \frac{z_1}{z_2}$$

Again, if we suppose z_2 to be constant, while y varies from y_1 to y_2, we shall have the value of x corresponding to y_2 and z_2, that is, x_2; and for that this equation holds:

$$② \ \frac{x'}{x_2} = \frac{y_1}{y_2}$$

Then by multiplying ① and ② we obtain

$$③ \ \frac{x_1}{x_2} = \frac{y_1 z_1}{y_2 z_2}$$

and as these are any two sets of corresponding values, the proof is general.

This theorem is of great importance in geometry.

CHAPTER XIII.

THE PROGRESSIONS.

338. A succession of algebraic terms which progress according to some definite law is called a **series.**

339. Among the simplest of series is that in which the terms progress by a common difference; thus

$$7;\ 10;\ 13;\ 16;\ 19;\ 22;\ 25;\ \text{etc.}$$

$$5x - 3y;\ 5x - y;\ 5x + y;\ 5x + 3y;\ 5x + 5y;\ \text{etc.}$$

In the first illustration above, the common difference is 3; the first term is 7; 3 is added to 7 once to give the second term, twice to give the third term, 3 times to give the fourth term, and so on.

In the second illustration the first term of the series is $5x - 3y$; the common difference is $2y$.

ARITHMETIC PROGRESSION.

340. A series in which the terms progress by a common difference is called an **Arithmetic Progression.***

In deducing formulæ for the laws of such a series, the letter a represents the first term, d the common difference, s the sum of n terms, l the nth term.

The first formula is seen at once :

$$\text{(I)} \quad l = a + (n - 1)d.$$

* Usually abbreviated A. P.; pronounced arithmet'ic progression.

The formula for the sum of n terms is obtained by first observing that the series can be written in reverse order by beginning with l and subtracting d; then for the sum of the series we have two expressions:

$$s = a + (a+d) + (a+2d) + (a+3d) + (a+4d) \ldots l$$
$$s = l + (l-d) + (l-2d) + (l-3d) + (l-4d) \ldots a$$

Adding these two we have

$$2s = (a+l) + (a+l) + (a+l) + (a+l) \ldots + (l+a)$$
$$= n(a+l)$$

$$\text{(II)} \quad s = \frac{n}{2}(a+l)$$

341. These two formulæ contain all the independent conditions that are imposed by the definition on the five unknown letters a, d, l, n, s. Hence we can determine any one of them, only if the three others are given.

Finding l and s.

342. Model A.—Find the 10th term in the series

$$35, 31, 27 \ldots$$
$$a = 35;\ d = -4;\ n = 10$$
$$l = 35 + 9(-4) = -1$$

Find the sum of the series, supposing it stops with the 13th term.

The 13th term is $l = 35 + 12(-4) = -13$

$$s = \tfrac{13}{2}(35 - 13) = 13(22) = 286$$

EXERCISE CLXI.

1. 11; 13; 15; . . .; 100th term.
2. 38; 30; 22; . . .; 18th term.
3. -100; -91; -82; . . .; 50th term.

4. -100; -91; -82; . . .; sum of 15 terms, beginning with the 3d term.

5. 83; 87; 91; . . .; to 20 terms; sum of the series.

6. $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{6}$; . . .; 10th term.

7. $\frac{1}{20}$; $\frac{1}{5}$; $\frac{7}{20}$; . . .; 28th term.

8. .001; .0055; .01; . . .; 23d term.

9. Sum of the first ten terms in $\frac{1}{20}$; $\frac{1}{5}$; $\frac{7}{20}$; . . .

10. Find the last term and the sum of the terms in an A. P. of 11 terms of which the first two are $a-x$ and $a-y$.

Eliminating with the Formulæ.

343. If any three of the five constants of an A. P. be given, substituting in the formulæ (I) and (II) gives two equations with two unknown letters, which can be determined by elimination.

Model B.—Given $a = 16$; $d = 4$; $s = 88$; find n.

(1) $l = 16 + (n-1)4 = 12 + 4n$

(2) $s = 88 = \dfrac{n}{2}(16 + l)$

(3) $88 = \dfrac{n}{2}(28 + 4n)$ — subst. (1) in (2)

(4) $4n^2 + 28n = 176$ — (3) $\times 2$

(5) $n^2 + 7n - 44 = 0$ — (4) $\div 4 - 44$

(6) $(n + 11)(n - 4) = 0$

$n = 4$; $n = -11$

Of these answers $n = 4$ is the only one that satisfies the conditions of the problem.

EXERCISE CLXII.

In the following examples find the constants that are not given:

	a	d	l	n	s
1.	5	2	25	....	....
2.	-10	3	....	38	....
3.	8	5	....	....	1572
4.	-3	....	787	80	....
5.	6	....	-34	....	-294
6.	11	....	....	36	-2754
7.	....	4	57	16	....
8.	....	-7	-36	....	-92
9.	....	....	397	200	19700
10.	....	11	....	13	182
11.	11	22	231	....	....
12.	$\frac{1}{9}$	$\frac{2}{9}$	....	100	....
13.	$\frac{7}{6}$	$\frac{1}{6}$	....	....	$977\frac{1}{2}$
14.	7	....	37	11	....
15.	9	....	-6	....	24
16.	$\frac{13}{27}$	....	....	5	$2\frac{1}{27}$
17.	....	$1\frac{1}{4}$	$25\frac{1}{2}$	17	....
18.	....	$\frac{1}{12}$	$\frac{2}{3}$	....	3
19.	....	....	$-9\frac{3}{4}$	21	$-99\frac{3}{4}$
20.	....	$-\frac{7}{8}$	....	25	$-181\frac{1}{4}$
21.	....	$\frac{1}{8}$	$2\frac{7}{8}$	....	$33\frac{3}{4}$

EXERCISE CLXIII.

Determine formulæ for each constant in terms of three others, as follows:

1. Find s; given a, d, and l.
   given a, d, and n.
   [given a, l, and n; formula II].
   given d, l, and n.
2. Find n; given a, d, and l.
   given a, d, and s.
   given a, l, and s.
   given d, l, and s.

3. Find l; [given a, d, and n; formula I].
given a, d, and s.
given a, n, and s.
given d, n, and s.

4. Find d; given a, l, and n.
given a, l, and s.
given a, n, and s.
given l, n, and s.

5. Find a; given d, l, and n.
given d, l, and s.
given d, n, and s.
given l, n, and s.

If one had a great many examples to work under any one of the 20 cases just given, it would pay to use one of the formulæ specially derived for that case. But in solving such as occur occasionally it is better to use the formulæ (I) and (II), substituting the given constants and deriving the others by elimination.

EXERCISE CLXIV.

In Arithmetic Progression,—

1. If the first term is 3 and the 5th term is 67, how many terms must be taken to add up 179?

2. When the 10th term is 6 and the 15th term is -2, find the sum of the first 8 terms.

3. When the 5th term is 10 and the 13th term is 101, find the sum of alternate terms, beginning with the 3d and ending with the 13th.

4. When the sum of the first 7 terms is equal to 3 times the next term, and the term after that is -2, what are the first 3 terms of the series?

5. When the sum of the first 3 terms is equal to half the sum of the first 7 terms, and greater by 14 than the sum of the second group of seven terms, what is the 16th term?

GEOMETRIC PROGRESSION.

344. A series in which the terms progress by a common ratio is called a **Geometric Progression.***

Such are the series,—

$$11; \quad 22; \quad 44; \quad 88; \quad 176; \ldots$$
$$24; \quad 8; \quad 2\tfrac{2}{3}; \quad \tfrac{8}{9}. \quad \tfrac{8}{27}; \ldots$$
$$x^3; \quad x^2y; \quad xy^2; \quad y^3; \quad \frac{y^4}{x}; \ldots$$

In each of these series the ratio of each term to the term preceding is the same throughout the series. The first term of the first series is multiplied by 2 to give the second term, by 2^2 to give the third term, by 2^3 to give the fourth term, and so on.

345. In deducing formulæ for G. P., we use the same letters as in A. P., except that instead of d for a common difference we have r for a common ratio.

The first formula for G. P. is seen at once:

$$\text{(III)} \quad l = ar^{n-1}$$

For the sum of n terms we have

$$\begin{aligned} s &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots + ar^{n-1} \\ &= a(1 + r + r^2 + r^3 + r^4 + r^5 + \ldots + r^{n-1}) \end{aligned}$$

The factor in parenthesis will be recognized as the quotient of $(1 - r^n) \div (1 - r)$; hence

$$\text{(IV)} \quad s = a\frac{1 - r^n}{1 - r} \equiv a\frac{r^n - 1}{r - 1}$$

Since $ar^n \equiv (ar^{n-1})r = lr$, we may write

$$\text{(IV)} \quad s = \frac{lr - a}{r - 1}$$

* Usually abbreviated G. P.

Finding l and s.

346. Model C. — Find the 10th term of the series $\frac{x^4}{y}$; x^3; x^2y; . . .

$$a = \frac{x^4}{y};\quad r = \frac{y}{x};\quad n = 10$$

$$l = \left(\frac{x^4}{y}\right)\left(\frac{y}{x}\right)^9 = \frac{x^4y^9}{x^9y} = \frac{y^8}{x^5}\ \textit{Ans.}$$

Model D.—Find the sum of the first six terms of the G. P. $\frac{8}{5}$; $\frac{8}{3}$; . . .

$$r = \tfrac{8}{3} : \tfrac{8}{5} = \tfrac{5}{3};\quad n = 6;\quad a = \tfrac{8}{5}$$

$$s = \frac{\frac{8}{5}\left(\frac{5}{3}\right)^5 - \frac{8}{5}}{\frac{5}{3} - 1} = \tfrac{12}{5}\left\{\left(\tfrac{5}{3}\right)^5 - 1\right\} = \tfrac{11528}{405}\ \textit{Ans.}$$

EXERCISE CLXV.

1. 2560; 1280; 640; . . .; 11th term.
2. 486; 162; 54; . . .; sum of first six terms.
3. $\frac{1}{15}$; $\frac{1}{5}$; $\frac{3}{5}$, . . .; 8th term.
4. 1; 2; 4; . . .; sum of first seven terms.
5. .0003; .003; .03; . . .; 10th term.
6. 3; -6; 12; -24; . . .; sum of first eleven terms.
7. x; 1; $\frac{1}{x}$; . . .; 20th term; kth term.
8. 5; $-\frac{1}{2}$; $+\frac{1}{20}$; . . .; sum of first five terms.
9. x^8; x^6y; x^4y^2; . . .; sum of first five terms.
10. $a^2 - x^2$; $a - x$; $\frac{a-x}{a+x}$; . . .; sum of first seven terms.

Eliminating with the Formulæ.

347. If any three of the five constants of a G. P. be given, substituting in the formulas (III) and (IV) gives two equations with two unknown letters, which can, theoretically at least, be determined by elimination, as in A. P.

As a matter of fact there are twelve cases where a solution is possible, and in these cases elimination is not necessary; in the other eight cases the equations obtained by substitution require either the use of logarithms or the solution of higher equations.

EXERCISE CLXVI.

In the following examples find the constants that are not given:

	a	r	l	n	s
1.	1215	$\frac{1}{3}$	....	6	....
2.	64	$-\frac{1}{2}$	....	....	1
3.	....	2	....	10	7161
4.	....	5	625	8	....
5.	....	$\frac{1}{2}$	....	7	$95\frac{1}{4}$
6.	....	$\frac{5}{2}$	$\frac{1}{32}$	....	.05155
7.	2	$\frac{3}{2}$	....	5	....
8.	2	5	781250	....	....
9.	5	....	160	6	....
10.	....	$-\frac{1}{2}$	$-\frac{1}{20}$	6	....
11.	5	....	327680	9	....
12.	$4\frac{1}{2}$	....	$\frac{32}{81}$	....	6
13.	-5	2	....	....	425
14.	.5	3	....	....	546.5
15.	....	$\frac{y}{x}$	y^7	8	....

Inserting Means.

348. The terms which intervene between the first and last terms in a finite series are called **means**. If the series is in A. P. these are called arithmetic means; if in G. P., geometric means.

When there is only one mean, it is known as **the arithmetic mean** (or **the geometric mean**, as the case may be), of the two quantities that serve as extremes.

When there are two or more means, they with the extremes make a finite series, in which the number of terms is two more than the number of means.

Model E.—Insert 5 geometric means between 16 and $182\frac{1}{4}$.

$$a = 16;\ l = 182\tfrac{1}{4};\ n = 7$$
$$182\tfrac{1}{4} = 16r^6$$
$$\tfrac{729}{64} = r^6$$
$$\tfrac{3}{2} = r$$

Series is 16, 24, 36, 54, 81, $\frac{243}{2}$, $\frac{729}{4}$.

Ans. 24; 36; 54; 81; $121\frac{1}{2}$.

EXERCISE CLXVII.

1. Insert 6 arithmetic means between 3 and 10.
2. Insert 6 arithmetic means between 3 and 13.
3. Insert 5 arithmetic means between 3 and 10.
4. Insert 7 arithmetic means between 3 and 10.
5. Find the arithmetic and geometric means between 15 and 900.
6. Insert 3 geometric means between 15 and 240.
7. Insert 3 geometric means between 15 and 1215.
8. Insert 4 geometric means between $\frac{1}{2}$ and 512.
9. Insert 6 geometric means between 14 and 1792.
10. Find the arithmetic and geometric means of the numbers $3\frac{1}{2}$ and $27723\frac{1}{2}$.

11. Insert 101 arithmetic means between -2 and 17.
12. Insert 9 arithmetic means between $a-b$ and $a+b$.
13. Insert 4 geometric means between x^5 and $32y^{10}$.
14. Insert 10 arithmetic means between $\frac{1}{10}$ and 10.
15. Insert x arithmetic means between $\dfrac{1}{x}$ and x.
16. Insert b arithmetic means between $a-b$ and $a+b$.
17. Insert $x-1$ geometric means between x and ax.
18. Insert 4 geometric means between $\dfrac{4}{3}$ and $\dfrac{3\sqrt{3}}{8}$.
19. Find the arithmetic mean and the geometric mean between a^3-3a-2 and a^3-4a^2+5a-2.
20. Insert 7 geometric means between $16a^{-4}$ and $\dfrac{a^4}{16}$.

Infinite Geometric Series.

349. When in a geometric progression the value of r is less than 1, the values of the successive terms become less and less, and the value of l, the nth term, may be made as small as we choose by taking n, the number of terms, large enough. In the formula

$$\text{(IV)}\quad s=\frac{lr-a}{r-1}$$

the term lr may be made smaller than any quantity that can be assigned, if we are permitted to take a sufficient number of terms; in other words, if the number of terms is unlimited, the more terms we take the nearer s comes to the value

$$s'=\frac{-a}{r-1}\equiv\frac{a}{1-r}$$

This is what is meant by the statement :

If the number of terms in a G. P. is infinite, and r is less than 1, then the sum of the series is

$$\text{(v)}\quad s' = \frac{a}{1-r}$$

350. This result may be confirmed by long division, from which we obtain

$$\frac{a}{1-r} \equiv a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + \dots$$

the number of terms being continued without limit.

351. Now this last identity can be tested with different values of r. Let us try $r = 2$.

$$\frac{a}{1-2}[\equiv -a] = a + 2a + 4a + 8a + 16a + 32a + 64a + \dots$$

The more terms we take of the infinite series on the right-hand side of this equation, the more their sum DIVERGES from the value of the expression that gave rise to the series. We come to this same curious conclusion whenever $r > 1$.

Again, suppose $r = \frac{1}{3}$.

$$\frac{a}{1-\frac{1}{3}}\left[\equiv \frac{3a}{2}\right] = a + \frac{a}{3} + \frac{a}{9} + \frac{a}{27} + \frac{a}{81} + \frac{a}{243} + \frac{a}{729} + \dots$$

The successive values of s, for different values of n, are shown in the following table :

Value of n	1	2	3	4	5	6
Value of s	a	$\frac{4a}{3}$	$\frac{13a}{9}$	$\frac{40a}{27}$	$\frac{121a}{81}$	$\frac{364a}{243}$
Difference between s and $\frac{3a}{2}$.	$\frac{a}{2}$	$\frac{a}{6}$	$\frac{a}{18}$	$\frac{a}{54}$	$\frac{a}{162}$	$\frac{a}{486}$

It is evident, then, that in this case, where $r = \frac{1}{3}$, the more terms we take, the closer their sum comes to the value $\dfrac{2a}{3}$.

352. An infinite series in which the sum of the first n terms approaches a definite limit as n is indefinitely increased is called a **convergent** series; if the sum of the first n terms does NOT approach a definite limit as n is indefinitely increased, the series is said to be **divergent.** This distinction is of the utmost importance in the further study of algebra.

Model F.—Find the sum of the infinite series

$$8, \quad 4, \quad 2, \ldots$$

$$s' = \frac{a}{1-r} = \frac{8}{1-\frac{1}{2}} = 16$$

Model G.—Find the value of the circulating decimal

$$3.7\dot{1}8\dot{2}^{*}$$

$$3.7\dot{1}8\dot{2} = 3.7 + .0\dot{1}8\dot{2}$$

$$.0\dot{1}8\dot{2} = .0182 + .0000182 + .0000000182 + \text{etc.}$$

$$= \frac{182}{10^4} + \frac{182}{10^7} + \frac{182}{10^{10}} + \cdots$$

Here $a = \dfrac{182}{10^4}$; $r = \dfrac{1}{10^3}$. Hence

$$s' = \frac{182}{10^4} : \left(1 - \frac{1}{10^3}\right) = \frac{.0182}{.999} = \frac{182}{9990}$$

Then $3.7 = 3\frac{6993}{9990}$.

$$3.7 + .0\dot{1}8\dot{2} = 3\tfrac{6993}{9990} + \tfrac{182}{9990} = 3\tfrac{1435}{1998} \textit{ Ans.}$$

* This is an abbreviation for 3.7182182182182, where the repetitions of 182 are continued indefinitely.

EXERCISE CLXVIII.

Find, wherever possible, the sum of the infinite series:

1. $54;\ 18;\ 6;\ \ldots$
2. $.7;\ .07;\ .007;\ \ldots$
3. $\frac{5}{9};\ \frac{1}{9};\ \frac{1}{45};\ \ldots$
4. $\frac{9}{4};\ \frac{3}{2};\ 1;\ \ldots$
5. $\frac{2}{3};\ 1;\ \frac{3}{2};\ \ldots$
6. $1;\ .2;\ .04;\ \ldots$
7. $2.7;\ .9;\ .3;\ \ldots$
8. $1.728;\ 1.44;\ 1.2;\ \ldots$
9. $7;\ 4.9;\ 3.43;\ \ldots$
10. $130;\ 26;\ 5.2;\ \ldots$

Find the values of the circulating decimals:

11. $5.\dot{7}$.
12. $5.\dot{7}\dot{3}$.
13. $5.\dot{7}3\dot{5}$.
14. $\dot{5}.7\dot{3}$.
15. $1.01101101101101\ \ldots$
16. $3.7\dot{3}\dot{6}$.
17. $8.1\dot{8}$.
18. $10.2\dot{0}\dot{1}$.
19. $\dot{7}0\dot{2}$.
20. $\dot{1}00.\dot{8}$.

21. The sum of an infinite series in G. P. is $33\frac{1}{3}$ per cent greater than the sum of the first two terms. What must be the value of r?

22. The sum of the fifth and sixth terms of a series in G. P. is 12 times the sum of the second and third terms. What is the ratio of the seventh term to the first?

23. The geometric mean of the first and fifth terms of a geometric series is 100; if the second term is 20, what is the fifth term?

24. The geometric mean of the second and sixth terms of a G. P. is 500; that of the fifth and seventh terms is 20. Find the series.

25. In a G. P. the arithmetic mean of the first two terms is 5 times the sum of the second and third terms. Find r.

26. Three numbers whose sum is 38 are in G. P.; if 1, 2, and 1 are added to them, in order, the results will be in A. P. Find the numbers.

27. In a certain G. P. the ratio of the third term to the

fifth is 7; what is the ratio of the first term to the seventh term?

28. How many terms of the series 37, 33, 29, . . . amount to 187? Explain the two answers.

29. By how much is the sum of the infinite series 256, 64, 16 . . . diminished if all terms after the fifth are cut off?

30. In finding the sum of the infinite series 9; -3; 1; . . . what is the error if we stop with the 10th term?

31. If in a given G. P. we find $r = .1$, find r for the series obtained by taking every fourth term of the given series.

32. Find an infinite series in G. P. such that each term is 4 times the sum of all the terms that follow it.

HARMONIC PROGRESSION.

353. A series in which the reciprocals of the terms are in arithmetic progression is called a **Harmonic Progression.***

Thus the numbers

$$60;\ 30;\ 20;\ 15;\ 12;\ 10$$

are in H. P. because their reciprocals

$$\tfrac{1}{60};\ \tfrac{2}{60};\ \tfrac{3}{60};\ \tfrac{4}{60};\ \tfrac{5}{60};\ \tfrac{6}{60}$$

are in A. P.

354. Problems in H. P. are solved by converting the given constants into the corresponding constants of arithmetic progression.

* Usually abbreviated H. P.

Model H.—Insert 3 harmonic means between 6 and 30.

This corresponds to an arithmetic progression in which $a = \frac{1}{6}$; $l = \frac{1}{30}$; $n = 5$.

$$\frac{1}{6} + 4d = \frac{1}{30}$$
$$5 + 120d = 1$$
$$120d = 44$$
$$d = -\frac{1}{30}$$

In A. P.: $\frac{1}{6}$; $\frac{2}{15}$; $\frac{1}{10}$; $\frac{1}{15}$; $\frac{1}{30}$

In H. P.: 6; $7\frac{1}{2}$; 10; 15; 30.

EXERCISE CLXIX.

1. What is the fifth term of a H. P. of which the second term is 5 and the fourth term is 10?

2. Find the sum of the first five terms of a H. P. of which the second and third terms are $\frac{1}{2}$ and $\frac{1}{3}$.

3. Write down the fifth and sixth terms of a H. P. of which the second term is 1 and the third term 2.

4. The sum of three terms of a H. P. is $1\frac{2}{9}$ and the last term is $\frac{2}{9}$. What are the other terms?

5. Insert four harmonic means between $12\frac{1}{2}$ and 75.

6. The first term of a harmonic series is 15015 and the fourth term is 5005. What is the seventh term?

7. Insert four harmonic means between 220 and 1540.

8. The first two terms of a harmonic series are -3 and $+3$; find the third term.

9. Insert four harmonic means between -3640 and 455.

10. Insert five harmonic means between 1 and 8; five geometric means; five arithmetic means.

The Three Means.

355. The harmonic mean of any two quantities is represented by H; the geometric mean by G; and the arithmetic

mean by A. They are determined by formulæ derived as follows, in each case from the definition of the series. Let x and y represent the extremes.

A; by definition of A. P., $A - x = y - A$

$$2A = x + y \qquad 1 + x + A$$

$$A = \frac{x+y}{2}.$$

G; by definition of G. P., $G : x = y : G$

$$xy = G^2 \qquad 1 \times xG$$

$$G = \sqrt{xy}$$

H; by definition of H. P., $\dfrac{1}{H} - \dfrac{1}{x} = \dfrac{1}{y} - \dfrac{1}{H}$

$$\frac{2}{H} = \frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy} \qquad 1 + \frac{1}{x} + \frac{1}{H}$$

$$H = \frac{2xy}{x+y} \qquad 1 : 2 \times 2$$

356. For distinguishing among the three types of series investigated in this chapter, the preceding formulæ furnish tests. Thus the three terms

$$60; \quad 70; \quad 84$$

are not in A. P.,—because $70 \neq \dfrac{60 + 84}{2}$

and not in G. P.,—because $70 \neq \sqrt{60 \times 84}$

but they are in H. P.,—because $70 = \dfrac{2 \times 60 \times 84}{60 + 84}$

EXERCISE CLXX.

1. Prove by this test that the three means A, G, H, for ANY two numbers always form a G. P.

2. Prove that in the proportion $x : y = p - x : y - p$, p is a harmonic mean between x and y.*

Identify the following series as A. P., G.P., or H. P.:

3. $\frac{1}{2}; \frac{1}{3}; \frac{1}{6}; \ldots$ **4.** $\frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \ldots$ **5.** $\frac{1}{2}; \frac{1}{3}; \frac{2}{9}; \ldots$

6. $3; 6; 9; \ldots$ **7.** $4; 2; 0; \ldots$ **8.** $7\frac{1}{2}; 10; 15; \ldots$

9. $3\frac{1}{3}; 5; 10; \ldots$ **10.** $2\frac{1}{2}; 5; 10; \ldots$ **11.** $65; 105; 273; \ldots$

12. $3\frac{\sqrt{2}}{5}; \frac{1}{5}\sqrt{90}; \sqrt{2}; \ldots$ **13.** $\frac{\sqrt{5}-2}{3}; \frac{1}{3}; \frac{\sqrt{5}+2}{3}; \ldots$

14. $\frac{2}{3}(\sqrt{8}+5); \frac{2(\sqrt{8}+1)}{7}; \frac{2}{5}(\sqrt{8}-3); \ldots$

15. $275; 385; 539; \ldots$ **16.** $924; 1309; 2244; \ldots$

17. $4788; 3800; 3150; \ldots$

18. Insert n arithmetic means between a and b and find the sum of the series thus formed.

19. If $a+b$, $b+c$, and $c+a$ are in G. P., prove that $\frac{a+b}{b+c} = \frac{c-a}{a-b}$.

20. If $y - x + z - x$; $x - y$; y are in G. P., prove that y; x; z also form a G. P.

21. There are m arithmetic means between 2 and 47, and the ratio of the 4th to the $(m-3)^{\text{th}}$ is $\frac{2}{5}$. Find m.

22. Find a number to which if 2, 8, and 17 be added the results are in G. P.

23. The arithmetic mean of two numbers exceeds the geometric by 24, and the geometric exceeds the harmonic by 30. Find the numbers.

* This property is sometimes used to define H. P.

24. The arithmetic mean of two numbers exceeds the geometric by a, and the geometric exceeds the harmonic by b. Find the arithmetic mean of the numbers.

25. Given A and G for any two numbers, find a formula for the numbers.

26. Given A and H for any two numbers, find a formula for the numbers.

27. Given H and G for any two numbers, find a formula for the numbers.

28. Find an A. P. whose first term is 7 and whose second, fifth, and tenth terms form a G. P.

CHAPTER XIV.

DETACHED COEFFICIENTS; BINOMIAL THEOREM.

Detached Coefficients in Multiplication.

357. In multiplying expressions which contain only one letter, after the expressions are arranged by descending powers of that letter, each term in the product, as in the multiplicand and multiplier, is determined as to its degree by its position in the expression. This fact leads to a very convenient method in Algebra, called the **method of detached coefficients**, which has several very important applications.

Model A.—*Arranging similar terms in columns:*

$(3x^3 - 2x^2 + 5x - 3)(2x^2 + 3x - 1)$.

$$\begin{array}{r}
3x^3 - 2x^2 + 5x - 3 \\
2x^2 + 3x - 1 \\
\hline
6x^5 - 4x^4 + 10x^3 - 6x^2 \\
+ 9x^4 - 6x^3 + 15x^2 - 9x \\
- 3x^3 + 2x^2 - 5x + 3 \\
\hline
6x^5 + 5x^4 + x^3 + 11x^2 - 14x + 3
\end{array}$$

The entire significance of this operation is preserved if we write only the coefficients, thus:

$$\begin{array}{r}
3 - 2 + 5 - 3 \\
2 + 3 - 1 \\
\hline
6 - 4 + 10 - 6 \\
9 - 6 + 15 - 9 \\
- 3 + 2 - 5 + 3 \\
\hline
6 + 5 + 1 + 11 - 14 + 3
\end{array}$$

In the same way the multiplication $(5x^3+3x^2-2x+2)(3x^2-2x+3)$ may be written:

Model B.

$$\begin{array}{rrrrrr}
5+ & 3- & 2+ & 2 & & \\
 & 3- & 2+ & 3 & & \\
\hline
15+ & 9- & 6+ & 6 & & \\
 & -10- & 6+ & 4- & 4 & \\
 & & 15+ & 9- & 6+ & 6 \\
\hline
15- & 1+ & 3+ & 19- & 10+ & 6
\end{array}$$

The answer being $15x^5-x^4+3x^3+19x^2-10x+6$.

In the following multiplication notice the significance of the zero coefficient in the result.

Model C.

$$(3x^4+5x^3-2x^2+7x-2)(5x^3+2x^2-7x-2).$$

$$\begin{array}{rrrrrrrr}
3+ & 5- & 2+ & 7- & 2 & & & \\
5+ & 2- & 7- & 2 & & & & \\
\hline
15+ & 25- & 10+ & 35- & 10 & & & \\
 & 6+ & 10- & 4+ & 14- & 4 & & \\
 & & -21- & 35+ & 14- & 49+ & 14 & \\
 & & & -6- & 10+ & 4- & 14+ & 4 \\
\hline
15+ & 31- & 21- & 10+ & 8- & 49+ & 0+ & 4
\end{array}$$

The answer being written :

$$15x^7+31x^6-21x^5-10x^4+8x^3-49x^2+4$$

Model D. $(3x^2+5)(2x^3+x-7)$.

$$\begin{array}{rrrrrr}
3+ & 0+ & 5 & & & \\
2+ & 0+ & 1- & 7 & & \\
\hline
3+ & 0+ & 5 & & & \\
 & & 3+ & 0+ & 5 & \\
 & & & -21+ & 0- & 35 \\
\hline
3+ & 0+ & 8- & 21+ & 5- & 35
\end{array}$$

$$3x^5+8x^3-21x^2+5x-35$$

The ordinary notation of numbers is by detached coefficients; thus 327 is really the same as $3t^2 + 2t + 7$ where $t = 10$; and 5002546 is the same as $5t^6 + 2t^3 + 5t^2 + 4t + 6$. It is complicated, however, by the fact that whenever the coefficient rises to the value of t the term becomes of higher degree.

EXERCISE CLXXI.

By detached coefficients multiply:

1. $2x^3 + 5x^2 + 7x - 2$ by $5x^2 - 7x + 3$.
2. $3x^2 - 5x + 1$ by $2x^4 + 3x^2 - 5x - 1$.
3. $(x^2 + 3x + 2)(2x^2 + 4x + 3)(x^2 - x - 1)$.
4. $(3x^2 + 5x - 3)^3$.
5. $(5x^4 - 2x^2 + 1)^2$.
6. $(x^6 - 5x^2 + 3)^2$.
7. $(2x^3 + 5x^2y + 7xy^2 - 2y^3)(5x^2 - 7xy + 3y^2)$.
8. $(3x^2 - 5xy + y^2)(2x^4 + 3x^2y^2 - 5xy^3 - y^4)$.
9. $(2x^2 - xy + 2y^2)^3$.
10. $(5x^4 - 3x^2y^2 + 2y^4)^2$.

Detached Coefficients in Division.

358. Model E.

$$(6x^5 + 5x^4y + x^3y^2 + 11x^2y^3 - 14xy^4 + 3y^5) : (3x^3 - 2x^2y + 5xy^2 - 3y^3).$$

$$\begin{array}{ll}
\text{Divisor.} & \underline{3 - 2 + 5 - 3} \\
\text{Quotient.} & \underline{2 + 3 - 1 } \\
\text{Dividend.} & 6 + 5 + 1 + 11 - 14 + 3 \\
 & \underline{6 - 4 + 10 - 6} \\
 & \phantom{6 - 4 +{}} 9 - 9 + 17 - 14 + 3 \\
 & \phantom{6 - 4 +{}} \underline{9 - 6 + 15 - 9} \\
 & \phantom{6 - 4 + 9 {}} - 3 + 2 - 5 + 3 \\
 & \phantom{6 - 4 + 9 {}} \underline{- 3 + 2 - 5 + 3} \\
 & \phantom{6 - 4 + 9 - 3 + 2 - 5 + {}} 0
\end{array}$$

The answer being $2x^2 + 3x - 1$.

EXERCISE CLXXII.

Use detached coefficients in dividing the following:

1. $(18x^4 - 45x^3 + 82x^2 - 67x + 40) \div (3x^2 - 4x + 5)$.
2. $(x^4 - 6x^3 + 9x^2 - 4) \div (x^2 - 3x + 2)$.
3. $(x^4 + x^3y - 8x^2y^2 + 19xy^3 - 15y^4) \div (x^2 + 3xy - 5y^2)$.
4. $(32x^4 + 5xy^3 - 81y^4) \div (2x + 3y)$.
5. $(x^5 - x^4 + x^3 - x^2 + x - 1) \div (x^3 - 1)$.
6. $(12 - 38x + 82x^2 - 112x^3 + 106x^4 - 70x^5) \div (7x^2 - 5x + 3)$.
7. $(9x + 3x^4 + 14x^3 + 2) \div (1 + 5x + x^2)$.
8. $(8x^8 - 16x^6 - 34x^4 + 32x^2 - 6) \div (2x^4 - 7x^2 + 3)$.
9. $(x^4 + 4x^3y + 6x^2y^2 + 5xy^3 + 2y^4) \div (x^2 + 3xy + 2y^2)$.
10. $(5x^4 + 2x^3y - 20x^2y^2 - 23xy^3 - 6y^4) \div (5x^2 + 7xy + 2y^2)$.

Detached Coefficients in H. C. F.

359. The method of detached coefficients often saves time in the process of H. C. F.

Model F. H. C. F. of

$2x^4 - 7x^3 + 12x^2 - 11x + 4.$	$3x^4 - 8x^3 + 5x^2 + 2x - 2.$	
(1) $2 - 7 + 12 - 11 + 4$	$3 - 8 + 5 + 2 - 2$	
(2) $3 - 8 + 5 + 2 - 2$		
(3) $6 - 21 + 36 - 33 + 12$	(1) $\times$ 3	
(4) $6 - 16 + 10 + 4 - 4$	(2) $\times$ 2	$1-2+1$ H.C.F.
*(5) $5 - 26 + 37 - 16$	(4) $-$ (3)	*Ans.*
*(6) $8 - 23 + 22 - 7$	(1) $+$ (4)	$[x^2 - 2x + 1]$
(7) $10 - 52 + 74 - 32$	(5) $\times$ 2	
(8) $2 - 29 + 52 - 25$	(7) $-$ (6)	
(9) $8 - 116 + 208 - 100$	(8) $\times$ 4	
(10) $93 - 186 + 93$	(6) $-$ (9)	
*(11) $1 - 2 + 1$	(10) $\div$ 93	
(12) $16 - 46 + 44 - 14$	(6) $\times$ 2	
(13) $11 - 20 + 7 + 2$	(12) $-$ (5)	
(14) $88 - 160 + 56 + 16$	(13) $\times$ 8	
*(15) $93 - 186 + 93$	(14) $+$ (5)	

Detached Coefficients in Elimination.

360. Space and perplexity may be saved in solving equations of more than two unknown letters by using a method of detached coefficients; that is, by omitting the letters and arranging the coefficients of the same letter always in the same column.

Model G.

$$\begin{aligned} 3x + 2y - 5z + w &= 30 \\ 2x - 3y + z - 4w &= -3 \\ 7x + 7y - 3z + 2w &= 93 \\ x - y + 4z + 3w &= 27 \end{aligned}$$

①	3	2	− 5	1	30	
②	2	− 3	1	4	− 3	
③	7	7	− 3	2	93	
④	1	− 1	4	3	27	
⑤	6	− 9	3	− 12	− 9	② × 3
⑥	13	− 2	*	− 10	84	③ + ⑤
⑦	8	− 12	4	− 16	− 12	② × 4
⑧	7	− 11	*	− 19	− 39	⑦ − ④
⑨	10	− 15	5	− 20	− 15	② × 5
⑩	13	− 13	*	− 19	15	① + ⑨
⑥	13	− 2		− 10	84	First
⑧	7	− 11		− 19	− 39	New
⑩	13	− 13		− 19	15	Set
⑪	6	− 2		*	54	⑩ − ⑧
⑫	26	− 4		− 20	168	⑥ × 2
⑬	13	9		− 1	153	⑫ − ⑩
⑭	130	90		− 10	1530	⑬ × 10
⑮	117	92		*	1446	⑭ − ⑥
⑪	6	− 2			54	Second
⑮	117	92			1446	New Set
⑯	276	− 92			2484	⑪ × 46
⑰	393				3930	⑮ + ⑯

From ⑰, $x = 10$
Subst. in ⑪, $60 - 2y = 54$; $y = 3$
Subst. in ⑥, $130 - 6 - 10w = 84$; $w = 4$
Subst. in ④, $10 - 3 + 4y + 12 = 27$; $y = 2$

Ans.

$$x = 10$$
$$y = 3$$
$$z = 2$$
$$w = 4$$

Coefficients of Powers.

361. The successive powers of $(a + b)$ may be calculated by detached coefficients. Thus we obtain:

$$\begin{array}{l}
1 + 1 \\
\underline{1 + 1} \\
1 + 1 \\
\underline{\phantom{1 +{}} 1 + 1} \\
1 + 2 + 1 \quad \text{(second power)}
\end{array}$$

$$\begin{array}{l}
\underline{\phantom{1+{}} 1 + 1 \phantom{{}+1}} \\
1 + 2 + 1 \\
\underline{\phantom{1 +{}} 1 + 2 + 1} \\
1 + 3 + 3 + 1 \quad \text{(third power)}
\end{array}$$

$$\begin{array}{l}
\underline{\phantom{1 + 3 +{}} 1 + 1} \\
1 + 3 + 3 + 1 \\
\underline{\phantom{1 +{}} 1 + 3 + 3 + 1} \\
1 + 4 + 6 + 4 + 1 \quad \text{(fourth power)}
\end{array}$$

Since the multiplier is always $1 + 1$, the set of coefficients for the next higher power can always be obtained from those of a given power by using them twice as a partial product, displaced as usual one term to the right. Thus

the calculation for successive powers would appear as follows:

3d	1+ 3+ 3+ 1
	1+ 3+ 3+ 1
4th	1+ 4+ 6+ 4+ 1
	1+ 4+ 6+ 4+ 1
5th	1+ 5+10+ 10+ 5+ 1
	1+ 5+ 10+ 10+ 5+ 1
6th	1+ 6+15+ 20+ 15+ 6+ 1
	1+ 6+ 15+ 20+ 15+ 6+ 1
7th	1+ 7+21+ 35+ 35+ 21+ 7+ 1
	1+ 7+ 21+ 35+ 35+ 21+ 7+ 1
8th	1+ 8+28+ 56+ 70+ 56+ 28+ 8+ 1
	1+ 8+ 28+ 56+ 70+ 56+ 28+ 8+ 1
9th	1+ 9+36+ 84+126+126+ 84+ 36+ 9+ 1
	1+ 9+ 36+ 84+126+126+ 84+ 36+ 9+ 1
10th	1+10+45+120+210+252+210+120+ 45+ 10+ 1
	1+10+ 45+120+210+252+210+120+ 45+10+ 1
11th	1+11+55+165+330+462+462+330+165+ 55+11+ 1
	1+11+ 55+165+330+462+462+330+165+55+11+1
12th	1+12+66+220+495+792+924+792+495+220+66+12+1

THE BINOMIAL THEOREM.

362. The coefficients of any power of the binomial $a + b$ are determinable by a famous rule, known as the **Binomial Theorem.**

In the first place these coefficients are the same whatever letters we use instead of a and b, and, in the enunciation of the rule here adopted, it will help us at first to get the coefficients of powers of $x + 1$.

363. In $(x + 1)^n$ the first term is x^n.

364. For the other coefficients:

Multiply the COEFFICIENT of any term by the INDEX and divide by the NUMBER OF THE TERM, to get the coefficient of the next term.

Model H.—In writing $(x + 1)^{13}$:

———the first term is x^{13}.

The coefficient of this term is 1;
the index is 13;
it is the first term.

$1 \times 13 \div 1 = 13$ (coefficient of the 2d term).

———the second term is $13x^{12}$.

The coefficient of this term is 13;
the index is 12;
it is term number 2.

$13 \times 12 \div 2 = 78$ (coefficient of the 3d term).

———the third term is $78x^{11}$.

The coefficient of this term is 78;
the index is 11;
it is term number 3.

$78 \times 11 \div 3 = 286$ (coefficient of the 4th term).

———and so on.

EXERCISE CLXXIII.

Write the first four terms of

1. $(x + 1)^{15}$. 2. $(x + 1)^{17}$. 3. $(x + 1)^{26}$.
4. $(x + 1)^{20}$. 5. $(x + 1)^{24}$

6. Write all the terms of $(x + 1)^{14}$.

7. Verify the coefficients obtained by the rule for $(x + 1)^4$ and $(x + 1)^8$.

8. Find the value of $(x + 1)^5$ when $x = 1$; also the value of $(x + 1)^n$.

9. What is the sum of the coefficients of any power of the binomial $x + 1$?

10. How many terms in the expansion * of $(x + 1)^n$?

11. The 10th term of $(a + b)^{19}$ is $92378a^{10}b^9$; find the coefficient of the next term.

* Where an algebraic expression of one term can be written out as a series of terms, that series is called the EXPANSION of the expression.

12. The 8th term of $(a+b)^{16}$ is $11440a^9b^7$; find the coefficient of the next term.

13. The 11th term of $(a+b)^{20}$ is $184756a^{10}b^{10}$; find the coefficient of the next term.

14. The 6th term of $(a+b)^{30}$ is $142506a^{25}b^5$; find the coefficient of the next term.

15. The 4th term of $(a+b)^{35}$ is $6545a^{32}b^3$; find the coefficient of the next term.

Symmetry of the Coefficients.

365. The coefficients obtained by this rule increase from the first term towards the middle, and decrease in the same way from the middle to the end, with exact symmetry; so that if the set of coefficients were turned end for end, the entire expression would be unchanged. If we write then the expansion of $(1+x)^n$ beginning at the END, with x^n, and understand by "the number of the term" the number COUNTING FROM THE END, the rule will give the same set of coefficients.

Now the expression $(a+b)^n$ has of course the same coefficients as $(x+1)^n$ and $(1+x)^n$. In applying the rule, if we count from the beginning, we must understand, by the words "the index," the index of a; and if from the other end, we must understand the index of b.

EXERCISE CLXXIV.

Find the first three and the last three terms of:

1. $(x+y)^{50}$. 2. $(a+b)^{60}$. 3. $(a+x)^{40}$.
4. $(x+a)^{55}$. 5. $(x+2)^{10}$.

Powers of Differences.

366. In the expansion of $(a-b)^n$ we must remember that $-b$ is a negative factor, and the terms will consequently be $+$ or $-$ according as an even or an odd number of factors b are contained in it; in other words, according as the index of b is even or odd.

EXERCISE CLXXV.

First five terms of:

1. $(a+b)^{20}$.
2. $(a+b)^{25}$.
3. $(a-b)^{18}$.
4. $(a-b)^{21}$.
5. $(a+b)^{23}$.

First three and last three terms of:

6. $(x+y)^{100}$.
7. $(x-y)^{20}$.
8. $(a-x)^{120}$
9. $(x-b)^{80}$.
10. $(a+y)^{110}$.

Irregular Coefficients.

367. It is evident that this rule will not apply to the numerical factors of the successive terms in expressions like $(2x-3)^5$.

Model I.

$$(2x-3)^5 \equiv 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243.$$

Here $32 \times 5 \div 1 \neq 240$; $240 \times 4 \div 2 \neq 720$; and so on. Such expressions are expanded by first writing the same power of a simple binomial as a pattern, and then substituting. As in this case:

$$(a-b)^5 \equiv a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

Now in the required expansion we must have $2x$ instead of a, and 3 instead of b; and making those substitutions we obtain the expansion as above.

EXERCISE CLXXVI.

In the same way expand:

1. $(3x-5)^6$.
2. $(2a-b^2)^8$.
3. $\left(x^2-\frac{y}{2}\right)^9$.
4. $(2x-\frac{1}{2})^7$.
5. $\left(3x+\frac{y}{3}\right)^4$.
6. $\left(\frac{a}{2}+\frac{2}{a}\right)^6$.
7. $(3a^2-2x)^7$.
8. $\left(1+\frac{x}{2}\right)^{10}$.
9. $\left(2+\frac{1}{x^3}\right)^7$.
10. $(p-5q)^5$.
11. $(3h-2k)^6$.

THE BINOMIAL THEOREM GENERALIZED.

368. In generalizing the binomial theorem, we obtain by the rule:

$$(x+1)^n \equiv x^n + nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^{n-4}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}x^{n-5}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6}x^{n-6} + \ldots$$

or, with two letters:

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^{n-4}b^4$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}a^{n-5}b^5 + \ldots$$

The coefficient for a term further on in the expansion becomes very unwieldy, and a generalized formula, for any term of any power, would be very hard to remember. To simplify the expression a new symbol is introduced.

Factorial n.

369. The factorial of any given number is the product of all the positive integers, beginning with 1 and ending with the given number.

The factorial of 5 is written $5!$ or $\underline{|5}$, and read "factorial five."

$$5! = 1.2.3.4.5 = 120$$
$$6! = 1.2.3.4.5.6 = 720$$

Higher factorials, like $20!$, give numbers that are inconveniently large in the ordinary notation. In figuring the values of such coefficients it is best to express them with prime factors, in the order of magnitude. Thus:

$$\begin{aligned}20! &= 1.2.3.2^2.5.2.3.7.2^3.3^2.2.5.11.2^2.3.13.2.7.3.5.2^4.17.2.3^2.19.2^2.5\\ &= 2^{18}.3^8.5^4.7^2.11.13.17.19\end{aligned}$$

370. In finding the values of expressions involving factorials, the work of calculation can often be shortened by factorization.

Model J.—In the expression $\dfrac{8!}{5!}$

the factors $5.4.3.2.1$ are common to numerator and denominator; hence

$$\frac{8!}{5!} = 8.7.6 = 336$$

Model K.

$$10! - 5! = 5!(10.9.8.7.6 - 1) = 120 \times 30239$$

EXERCISE CLXXVII.

Find the values of the following expressions:

1. $\dfrac{11!}{9!}$. **2.** $\dfrac{11! - 7!}{6!}$. **3.** $\dfrac{3!(8! - 6!)}{5!}$.

4. $\dfrac{10!}{5!\,5!}$. **5.** $\dfrac{11!}{3!\,8!} - \dfrac{11!}{2!\,9!}$. **6.** $\dfrac{8!}{5!} - \dfrac{7!}{5!}$.

7. $\dfrac{20! - 18!}{187(10!\,8!\,5!)}$. **8.** $\dfrac{100! - 99!}{3(99!)}$.

9. $\dfrac{87!}{29(86!)} - 1$. **10.** $\dfrac{13! - 11!}{55(9!)}$.

The symbol $8!$ *is in some modern books largely replaced by* $\underline{|8}$; *the symbols have the same meaning, and both are in good use. They are used indifferently in the following:*

11. What factors must be multiplied into $8 \cdot 7 \cdot 6$ to give $\underline{|8}$? [Express in briefest form.]

12. What factors must be multiplied into $10 \cdot 9 \cdot 8$ to give $\underline{|11}$?

13. What factors must be multiplied into $20 \cdot 19 \cdot 17$ to give $\underline{|20}$?

14. What factors must be multiplied into $\underline{|n}$ to give $\underline{|(n+2)}$?

15. What factors must be multiplied into $n(n-1)(n-2)$ to give $\underline{|n}$?

Find the value of n in the following equations:

16. $\dfrac{n!}{(n-1)!} = 7.$

17. $\dfrac{n!}{(n-2)!} = 72.$

18. $\dfrac{(n+1)!}{(n-1)!} = 110.$

19. $\dfrac{(n-3)!}{(n-5)!} = 42.$

20. $\dfrac{105(n-4)!}{(n-3)!} = 2n+5.$

21. $\sqrt{\dfrac{\underline{|7}\,\underline{|n-3}\,(n-1)}{\underline{|n}}} = \dfrac{1}{3}.$

22. $\dfrac{\underline{|n+2}}{\underline{|4}\,\underline{|n-2}} : \dfrac{\underline{|n+1}}{\underline{|5}\,\underline{|n-4}} = \dfrac{9}{4}.$

23. $\dfrac{4!(n-2)!}{(n-6)!} : \dfrac{(n-3)!}{46!(n-5)!} = 2\dfrac{48!}{47}.$

24. $\dfrac{(n+4)(n-5)!}{(n-4)!} - \dfrac{(n-4)(n+3)!}{(n+4)!} = \dfrac{5}{6}.$

25. $\dfrac{n!(n-2)!}{[(n-1)!]^2} - \dfrac{(n!)^2}{(n+1)!(n-1)!} = .225.$

For Any Term of Any Power.

371. To generalize the binomial theorem, so as to be able to write any term of any power directly from a formula, let us first write

$$(x+y)^n \equiv x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + a_3x^{n-3}y^3 + a_4x^{n-4}y^4 + \ldots$$

where a_1, a_2, a_3, a_4, etc., represent the successive coefficients, as obtained by the rule.

372. In each term there is a number (in the third term, for example, it is 2) which appears in three places:

it is the index of y;

it is subtracted from n to give the index of x;

it is the suffix of a.

This number is called the **modulus** of the term and is represented by k;

———in the first term $k = 0$,

———in the second term $k = 1$,

———in the third term $k = 2$,

———in the fourth term $k = 3$,

and so on; in general,—

373. The modulus of any term is one less than the number of the term.

374. We may now state, and will hereafter prove,

THE BINOMIAL FORMULA:

$$a_k = \frac{n!}{k!(n-k)!}.$$

Model L.—The 7th term of $(x + y)^{11}$ is

$$\frac{11!}{6!\ 5!}x^5y^6.$$

This reduces $\dfrac{11.10.9.8.7}{1.2.3.4.5} = 2.3.7.11 = 462$ as previously found.

Model M.—The 4th term of $(x - y)^{20}$ is

$$-\frac{20!}{3!\ 17!}x^{17}y^3.$$

The coefficient reduces to $\dfrac{20\,.\,19\,.\,18}{1\,.\,2\,.\,3} = 20\,.\,19\,.\,3 = 1140.$

Model N.—The 13th term of $(a-b)^{25} = +\frac{25!}{12!\,13!}a^{13}b^{12}$.

$$\frac{25!}{12!\,13!} = \frac{25.24.23.22.21.20.19.18.17.16.15.14}{1.2.3.4.5.6.7.8.9.10.11.12} = 2^2.5^2.7.17.19.23.$$

Ans. $2^2.5^2.7.17.19.23a^{13}b^{12}$.

EXERCISE CLXXVIII.

By means of the formula write:

1. $(a-b)^{22}$; 9th term.
2. $(x-y)^{27}$; 7th term.
3. $(x-y)^{26}$; 6th term.
4. $(p-q)^{19}$; 11th term.
5. $(h-k)^{18}$; 8th term.
6. $(a+b)^{24}$; 12th term.
7. $(x+y)^{30}$; 6th term.
8. $(x-y)^{28}$; 10th term.
9. $(h+k)^{25}$; 14th term.
10. $(s-t)^{23}$; 13th term.
11. $(a-\sqrt{2})^{23}$; 10th term.
12. $\left(a^2-\frac{1}{a}\right)^{28}$; 8th term.
13. $\left(a-\frac{1}{\sqrt{a}}\right)^{27}$; 7th term.
14. $\left(x-\frac{\sqrt{x}}{3}\right)^{20}$; 12th term.
15. $\left(\frac{x}{y^2}-y\right)^{19}$; 9th term.
16. $\left(\frac{x}{\sqrt{2y}}-\frac{1}{\sqrt{x}}\right)^{25}$; 13th term.
17. $\left(\frac{a}{\sqrt{2x}}-\frac{\sqrt{2x}}{6a}\right)^{31}$; 7th term.
18. $\left(\frac{3x^2}{a}-\frac{a}{3x}\right)^{29}$; 11th term.
19. $\left(2x^2y-\frac{1}{3y\sqrt{x}}\right)^{26}$; 15th term.
20. $\left(\sqrt[3]{x}-\sqrt{\frac{y}{x}}\right)^{24}$; 14th term.

375. According to the Binomial Formula,

$$\text{in } (a+b)^n, \quad a_k = \frac{n!}{k!(n-k)!}$$

$$\text{in } (a+b)^{n+1}, \quad a_k = \frac{(n+1)!}{k!(n+1-k)!}$$

Now if we could prove,-

First: that the formula gives the correct coefficients for the first few powers of $(a+b)$; and,

Secondly: that if the formula gives the correct coefficients for any one power of $(a+b)$ it must also give the correct coefficients for the next higher power;

then we should have to admit that the formula was valid for all powers of $(a+b)$.

Proof of the Binomial Formula.

376. Referring to the calculation of successive powers of a binomial, earlier in this chapter, where the method of detached coefficients was used, it is easy to see that in passing from one power of $(a+b)$ to the next higher power, the following rule holds:

In any power of $(a+b)$, starting with a term of given modulus, if we add to its coefficient the preceding coefficient, we obtain **the coefficient of the term having the same modulus** in the next higher power of $(a+b)$.

377. In $(a+b)^n$, therefore, if the formula holds good, the coefficient of the term of modulus k is $\frac{n!}{k!(n-k)!}$, and the coefficient of the preceding term, whose modulus is $k-1$, is $\frac{n!}{(k-1)!(n-k+1)!}$.

The coefficient of the term of modulus k, in $(a+b)^{n+1}$, would consequently be

$$\frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}$$

This reduces:

$$\frac{n!}{(k-1)!(n-k)!}\left\{\frac{1}{n-k+1}+\frac{1}{k}\right\}$$

$$\equiv\frac{n!}{(k-1)!(n-k)!}\cdot\frac{n+1}{k(n+1-k)}\equiv\frac{(n+1)!}{k!(n+1-k)!}$$

which is the coefficient that the formula would give.

378. Thus we have shown that, if the binomial formula holds for $(a+b)^n$, it holds also for $(a+b)^{n+1}$. Now the formula does hold for $(a+b)^2$, $(a+b)^3$, and so on, as far as we choose to test it among the low powers of $a+b$. According to our proof, if it holds for $(a+b)^3$, it holds for $(a+b)^4$; and if for $(a+b)^4$, then also for $(a+b)^5$; then also for $(a+b)^6$, and so on, step by step, through all powers of $(a+b)$ for which the indices are positive integers.*

From the Formula to the Rule.

379. In the expansion of $(x+y)^n$ the term of modulus k is, by the formula, $\frac{n!}{k!(n-k)!}x^{n-k}y^k$; and the next term, which has for a modulus $k+1$, is

$$\frac{n!}{(k+1)!(n-k-1)!}x^{n-k-1}y^{k+1};$$

* This method of proof is called MATHEMATICAL INDUCTION.

the numerical coefficients of these terms, being represented by a_k and a_{k+1} respectively, are in the ratio

$$\frac{a_{k+1}}{a_k} = \frac{n!}{(k+1)!(n-k-1)!} \div \frac{n!}{k!(n-k)!}$$
$$\equiv \frac{k!(n-k)!}{(k+1)!(n-k-1)!} \equiv \frac{n-k}{k+1}$$

Whence we obtain $a_{k+1} = \dfrac{n-k}{k+1}a_k$; and this agrees with the form of the theorem first enunciated, because, in the term of modulus k, $n-k$ is the index of x, and $k+1$ is the number of the term.

Negative and Fractional Indices.

380. Thus far we have considered the binomial theorem only for cases where n was a whole number, and a positive number. It holds also, with certain exceptions, for negative and fractional values of n; but in such cases it always gives an infinite series.

In such cases also we must alter the binomial formula; for if n is fractional, or negative, $n!$ and $(n-k)!$ become meaningless symbols.

381. The coefficient a_k may ALWAYS be written with $k!$ for its denominator, and with the product of k factors for its numerator,—these k factors being

$$n(n-1)(n-2)(n-3)\ldots(n-k+1)$$

That is: $\quad a_k = \dfrac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$

382. We may verify the formula in some cases as follows:

Model 0. $\sqrt{1-x} \equiv (1-x)^{\frac{1}{2}}$; expand in a series.

By the formula: $(1-x)^{\frac{1}{2}} \equiv 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \ldots,$

Otherwise:

$$\begin{array}{r|l}
\sqrt{1 - x} & 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \dots \\
1 & \\
\hline
- x & -\frac{x}{2} \\
- x + \frac{x^2}{4} & \\
\hline
- \frac{x^2}{4} & 2 - x - \frac{x^2}{8} \\
- \frac{x^2}{4} - \frac{x^3}{8} - \frac{x^4}{64} & \\
\hline
\frac{x^3}{8} + \frac{x^4}{64} & 2 - x - \frac{x^2}{4}
\end{array}$$

Model P.—Expand $(1 + x)^{-2} \equiv \dfrac{1}{1 + 2x + x^2}$.

By the formula: $(1+x)^{-2} \equiv 1-2x+3x^2-4x^3+5x^4- \dots$

Otherwise:

$$\begin{array}{l}
1 + 2x + x^2 \\
\hline
1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots \\
\hline
1 \\
1 + 2x + x^2 \\
\hline
- 2x - x^2 \\
- 2x - 4x^2 - 2x^3 \\
\hline
+ 3x^2 + 2x^3 \\
+ 3x^2 + 6x^3 + 3x^4 \\
\hline
\quad - 4x^3 - 3x^4 \\
\quad - 4x^3 - 8x^4 - 4x^5 \\
\hline
\quad\quad 5x^4 + 4x^5 \\
\hline
\end{array}$$

Model Q.—Expand $(x + y)^{-2}$.

$$(x+y)^{-2} \equiv x^{-2} + (-2)x^{-3}y + \frac{(-2)(-3)}{2\,!}x^{-4}y^2 + \frac{(-2)(-3)(-4)}{3\,!}x^{-5}y^3 + \dots$$

$$\equiv x^{-2} - 2x^{-3}y + 3x^{-4}y^2 - 4x^{-5}y^3 + \dots$$

$$\equiv \frac{1}{x^2} - \frac{2y}{x^3} + \frac{3y^2}{x^4} - \frac{4y^3}{x^5} + \dots$$

Model R.—Find the 11th term of $(x - y)^{-\frac{4}{3}}$.

$$\frac{\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(-\frac{10}{3}\right)\left(-\frac{13}{3}\right)\left(-\frac{16}{3}\right)\left(-\frac{19}{3}\right)\left(-\frac{22}{3}\right)\left(-\frac{25}{3}\right)\left(-\frac{28}{3}\right)\left(-\frac{31}{3}\right)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10} x^{-\frac{4}{3}-10}(-y)^{10}$$

$$\equiv \frac{1}{3^{10}} \cdot \frac{4\,.\,7\,.\,10\,.\,13\,.\,16\,.\,19\,.\,22\,.\,25\,.\,28\,.\,31}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10} x^{-\frac{34}{3}} y^{10}$$

$$\equiv \frac{2^2.5.7.11.13.19.31}{3^{14}} \cdot \frac{y^{10}}{x^{11}\sqrt[3]{x}} \equiv \frac{2^2.5.7.11.13.19.31}{3^{14}} \cdot \frac{y^{10}\sqrt[3]{x^2}}{x^{12}}$$

EXERCISE CLXXIX.

Expand by the binomial theorem, to four terms:

1. $(x+y)^{-3}$. **2.** $(x+1)^{\frac{2}{3}}$. **3.** $(a+b)^{-1\frac{1}{2}}$. **4.** $(x-y)^{-5}$.

5. $(1-x)^{\frac{3}{4}}$. **6.** $(x+2)^{-3}$. **7.** $(3x+1)^{\frac{2}{3}}$. **8.** $(2a+3b)^{-1\frac{1}{2}}$.

9. $(x^2-\frac{1}{2})^{-5}$. **10.** $\left(\frac{a}{3}-\frac{4}{a}\right)^{\frac{3}{4}}$

Expand and verify, to four terms:

11. $(x+y)^{-1}$. **12.** $(1+x)^{\frac{1}{2}}$. **13.** $(2-x)^{-3}$. **14.** $(1+x)^{\frac{3}{2}}$.

15. $(1-x)^{-\frac{1}{2}}$. **16.** $(1-x)^{-1}$.

17. In the series obtained by the binomial theorem for $(1 - x)^{-1}$ substitute $x = \frac{1}{2}$; substitute also in $(1-x)^{-1} \equiv \dfrac{1}{1 - x}$ and compare the results; verify by the theory of geometrical progression.

18. In the same series substitute $x = 2$; is this series as clearly equal to the expression from which it arose? Is the series obtained by division any more accurate? Why does the theory of geometrical progression fail us here?

19. Find the 8th term in $(x + y)^{\frac{5}{3}}$.

20. Find the 10th term in $(1 - x)^{\frac{5}{2}}$.

CHAPTER XV.

LOGARITHMS.

383. The work of calculating numerical products, quotients, powers, and roots is much simplified by expressing all numbers as powers of some single number; all these operations then become addition, subtraction, multiplication, and division of the indices. In this chapter it is shown that such a simplification is practicable.

Table of Powers of 10.

384. By successive applications of square and cube root the following fractional powers of 10 can be calculated to three places of decimals: $10^{\frac{1}{16}}$; $10^{\frac{1}{8}}$; $10^{\frac{1}{6}}$; $10^{\frac{3}{16}}$; $10^{\frac{1}{4}}$; $10^{\frac{5}{16}}$; $10^{\frac{1}{3}}$; $10^{\frac{3}{8}}$; $10^{\frac{7}{16}}$; $10^{\frac{1}{2}}$; $10^{\frac{9}{16}}$; $10^{\frac{5}{8}}$; $10^{\frac{2}{3}}$; $10^{\frac{11}{16}}$; $10^{\frac{3}{4}}$; $10^{\frac{13}{16}}$; $10^{\frac{5}{6}}$; $10^{\frac{7}{8}}$; $10^{\frac{15}{16}}$.

Thus we get the accompanying table, in which, for example, $10^{\frac{1}{16}}$ (or $10^{.0625}$) is shown to be $= 1.154$; $10^{\frac{13}{16}}$ (or $10^{.8125}$) $= 6.494$; etc.

Indices.		Powers.
$\frac{1}{16}$	.0625	1.154
$\frac{1}{8}$	.1250	1.334
$\frac{1}{6}$	.1667	1.468
$\frac{3}{16}$	.1875	1.540
$\frac{1}{4}$	.2500	1.778
$\frac{5}{16}$	.3125	2.048
$\frac{1}{3}$	.3333	2.154
$\frac{3}{8}$	.3750	2.371
$\frac{7}{16}$	.4375	2.738
$\frac{1}{2}$	.5000	3.162
$\frac{9}{16}$	.5625	3.652
$\frac{5}{8}$	.6250	4.217
$\frac{2}{3}$	.6667	4.641
$\frac{11}{16}$	.6875	4.870
$\frac{3}{4}$	.7500	5.623
$\frac{13}{16}$	.8125	6.494
$\frac{5}{6}$	.8333	6.813
$\frac{7}{8}$	.8750	7.499
$\frac{15}{16}$	.9375	8.660

385. This table can be extended (by other means) till every number **between 1 and 10**, to two or three or even more decimal places, has been exhibited as a power of 10.

Such a table is called a table of logarithms, or, more exactly, a **table of common logarithms**; for other tables might be constructed, in which numbers would be exhibited as powers of some root other than 10.

In every such system,—

386. The common root of all the numbers is called the **base.**

387. The indices are called **logarithms.**

Using the Tables.

388. In four-place tables, the first two figures of each number are given in the column marked N; and the third figure is given at the head of one of the ten columns of logarithms. Thus for the number 4.27 we should find the first two figures 42 in the N column, and on the same line, in the column marked 7, we should find 6304. [Decimal points are omitted.] We conclude, then, that $4.27 = 10^{.6304}$; or in other words, the logarithm of 4.27 is .6304.

Similarly, if we had the logarithm .5092, to find the number which corresponds to it, we first look up 5092 among the ten columns of logarithms, notice that it is on a line with 32 in the column N and has 3 at the head of its own column, and so conclude that 5092 is the logarithm of 3.23.

EXERCISE CLXXX.

Find:

1. log 3.73; log 4.32
2. log 5.85; log 3.54
3. log 2.97; log 7.29
4. log 1.73; log 6.4
5. log 8.9; log 9

Find numbers whose logarithms are:

6. .4771; .8597 **7.** .9814; .7980 **8.** .1987; .4518

9. .5024; .4886 **10.** .8645; .9562

Interpolation.

389. Model A.—Find log 2.573.

This number is $\frac{3}{10}$ of the way from 2.57 to 2.58; and as we find from the table that the log 2.57 is .4099 and log 2.58 is .4116, we assume that log 2.573 is $\frac{3}{10}$ of the way from .4099 to .4116.

The difference between two successive logarithms in the table is called the **tabular difference** and is represented by D; and D has different values in different parts of the table.

The difference between the required logarithm and the NEAREST TABULAR LOGARITHM is represented by d.

The assumption made above is that the difference of two numbers, in the same part of the table, is proportional to the difference of their logarithms. This assumption is not absolutely correct, but leads to no serious error.

$$\frac{2.573 - 2.57}{2.58 - 2.57} = \frac{d}{D}$$

$$.3 = \frac{d}{D}$$

$$d = (.3)D = (.3)(17) = 5; \ \log 2.573 = .4104$$

Model B.—Find log 3.736.

$$\begin{aligned} \log 3.74 &= .5729 \\ \log 3.73 &= .5717 \\ \hline D &= \quad 12 \end{aligned} \qquad \begin{aligned} \log 3.74 &= .5729 \\ d &= \quad\ \ 5 \\ \hline \log 3.736 &= .5724 \end{aligned}$$

$$d = (.4) \times 12 = 4.8$$

In most tables of logarithms the values of $\frac{1}{10}D$, $\frac{2}{10}D$, $\frac{3}{10}D$, etc., are given in the margin opposite the portion of the table for which they are correct, thus making it possi-

ble to determine the logarithm mentally. These multiples of $\frac{D}{10}$ are called **Proportional Parts.**

Model C.—Find the number whose logarithm is .7470.

$$\begin{aligned} \log 5.58 &= .7466 \\ \log 5.59 &= .7474 \\ D &= \quad\; 8 \\ d &= \quad\; 4 \\ \frac{d}{D} = \frac{4}{8} &= .5 \end{aligned}$$

Ans. $\log 5.585 = .7470.$

The table of proportional parts can also be used in this process.

EXERCISE CLXXXI.

Find:

1. $\log 1.832$ 2. $\log 2.471$ 3. $\log 5.234$
4. $\log 4.788$ 5. $\log 7.323$

Find numbers whose logarithms are:

6. .2835 7. .3048 8. .9873 9. .5747 10. .4891

Characteristics.

390. Although the tables give only the logarithms of numbers between 1 and 10, the logarithms of numbers below 1 and above 10 can readily be obtained also, remembering that the logarithm of a number is its index considered as a power of 10.

Model D.—Thus $395.2 = 100 \times 3.952 = 10^2 \times 10^{.5968}$, since log 3.952 is, by the table, .5968.

$$10^2 \times 10^{.5968} = 10^{2.5968}$$

Hence $\qquad \log 395.2 = 2.5968$

Every number may in the same way be separated into two factors, of which one is a number between 1 and 10, and the other is a power of 10, positive or negative. Thus

$$385400 = 3.854 \times 10^5$$

$$.0853 = 8.53 \div 100 = 8.53 \times 10^{-2}.$$

Model E.—Find log .003987

$$.003987 = 3.987 \div 1000 = 3.987 \times 10^{-3}$$

$$= 10^{-3} \times 10^{.6006}$$

In practice this subtraction is not carried out, but indicated in a contracted form by writing the negative sign over the characteristic, thus: log $.003987 = \overline{3}.6006$.

391. The integral part of a logarithm is called the **CHARACTERISTIC**, and the rest of the logarithm, comprising all the figures on the right of the decimal point and constituting a proper decimal fraction, is called the **MANTISSA.**

392. The characteristic may be found by counting off the places from the first significant figure of the number to the units place.

Numbers between 0 and 1 will have negative characteristics; negative numbers have no logarithms.

EXERCISE CLXXXII.

Find:

1. log 283.5
2. log 30.13
3. log 78.45
4. log .07832
5. log 11.037

Find numbers whose logarithms are:

6. 5.8372
7. $\overline{1}.1982$
8. 2.6570
9. $\overline{3}.6670$
10. 3.4825

Augmented Logarithms.

393. The confusion which may easily be expected to arise with the use of negative characteristics is avoided by adding 10 to every logarithm complicated by such a characteristic. Thus log .003987 would appear in calculation as 7.6006X; the X indicating an excess of 10 to be subtracted from the result.

Calculation by Logarithms.

394. Since powers are multiplied or divided by adding or subtracting their indices, so any numbers may be multiplied or divided by adding or subtracting their logarithms.

395. In the same way roots and powers are obtained by dividing or multiplying logarithms by the required index.

Model F.- Calculate $\dfrac{(398.7)(.0983)(9.837)}{(98.07)(38.9)(.00783)(7.38)}$

$$\begin{array}{ll} & \log 98.07 \;\;= 1.9915 \\ \log 398.7 = 2.6006 & \log 38.9 \;\;\;= 1.5899 \\ \log .0983 = 8.9926\,\text{X} & \log .00783 = 7.8938\,\text{X} \\ \log 9.837 = 0.9929 & \log 7.38 \;\;= 0.8681 \\ \hline \qquad\qquad 12.5861\,\text{X} & \qquad\qquad 12.3433\,\text{X} \end{array}$$

$$\log \text{ans.} = 2.5861 - 2.3433 = .2428$$
$$= \log 1.749 \qquad\qquad 1.749 \; \textit{Ans.}$$

EXERCISE CLXXXIII.

Calculate:

1. $(28.3)(.0587)(8.93)(1.1354)$.
2. $(.0057)(.00342)(.007893)(8496000)$.
3. $(87.6)(83.9)(.000786)(508)$.
4. $(98.7)(79.8)(9780)(.00789)(8.97)$.
5. $(8.732)^5$. 6. $\sqrt[5]{8732}$.

7. $(.00768)(53.42)^3 : 1.358$.

8. $(\sqrt[3]{867 : 35.84})(1.537)$.

9. $(34020)^2(.0000842)$.

10 $\sqrt[3]{(63.8)(47.02)(25.37)}$.

Cologarithms.

396. The computation of expressions involving division is much simplified by the use of **cologarithms**. The cologarithm (more exactly, the arithmetical complement of the logarithm) of a number is **10 minus its logarithm**. It is computed by subtracting each digit of the logarithm from 9 except the last, and that from 10. Thus log 398.7 = 2.6006 and colog 398.7 = 7.3994 X.

397. If of two logarithms l_1 and l_2 the second were to be subtracted from the first, the same result, $l_1 - l_2$, could be obtained by adding to l_1 the cologarithm derived from l_2, WITH AN EXCESS OF 10 ON ACCOUNT OF THE COLOGARITHM; that is, $l_1 - l_2 \equiv (l_1 + 10 - l_2) - 10$.

Thus in computing the expression in Model F we have performed two additions and one subtraction; the use of cologarithms would reduce this to one addition, as follows:

log 398.7 + log .0983 + log 9.837 − log 98.07 − log 38.9
− log .00783 − log 7.38 becomes

log 398.7 + log .0983 + log 9.837 + 10 − log 98.07
+ 10 − log 38.9 + 10 − log .00783 + 10 − log 7.38; or
log 398.7 + log .0983 + log 9.837 + colog 98.07
+ colog 38.9 + colog .00783 + colog 7.38

398. The cologarithm of .00783 DOES NOT IMPLY AN EXCESS OF 10, because 10 has already been added to the logarithm on account of the negative characteristic; that is,

$$10 - (10 + \log) = -\log$$

With these changes the calculation of Model F becomes:

Model G.

$$
\begin{array}{rl}
\log 398.7 & = 2.6006 \\
\log .0983 & = 8.9926 \text{ X} \\
\log 9.837 & = 0.9929 \\
1.9915 \text{ colog } 98.07 & = 8.0085 \text{ X} \\
1.5899 \text{ colog } 38.9 & = 8.4101 \text{ X} \\
7.8938 \text{ X colog } .00783 & = 2.1062 \\
0.8681 \text{ colog } 7.38 & = 9.1319 \text{ X} \\
\hline
\log \text{ ans.} & = 40.2428 \text{ XXXX} \\
 & = .2428
\end{array}
$$

Ans. 1.749.

The Logarithmic Schedule.

399. One of the most important things in all logarithmic computation is ORDERLY ARRANGEMENT; and the pupil is recommended to arrange his work in advance, by writing a **schedule** of the logarithms before opening his tables.

Model H. $\sqrt[3]{\dfrac{(3.87)(3.087)(38.07)}{30.08}}.$

The schedule would be made and filled out as follows (wherever a cologarithm is called for, the logarithm from which it is derived is written on the left of the schedule):

$$
\begin{array}{rl}
\log 3.87 & = 0.5877 \\
\log 3.087 & = 0.4896 \\
\log 38.07 & = 1.5806 \\
1.4783 \text{ colog } 30.08 & = 8.5217 \text{ X} \\
\hline
\log \text{ fraction} & = 11.1796 \text{ X} \\
 & = 1.1796 \\
\log \sqrt[3]{\text{fraction}} & = .3932
\end{array}
$$

Ans. 2.473.

In-radius of a Triangle.

400. If the three sides of a triangle are represented by a, b, and c, the radius of the inscribed circle is determined by the formula $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, where

$$s = \frac{a+b+c}{2}.$$

Model I.—Compute r for the triangle $a = 253$, $b = 260$, $c = 315$.

$$\begin{array}{rrll}
a = 253 & & \log(s-a) & = 2.2068 \\
b = 260 & & \log(s-b) & = 2.1875 \\
c = 315 & & \log(s-c) & = 1.9956 \\
\hline
2s = 828 & 2.6170 & \text{colog } s & = 7.3830 \text{ X} \\
s = 414 & & \log r^2 & = 3.7729 \\
s - a = 261 & & \log r & = 1.8864 \\
s - b = 154 & & r & = 76.98 \\
s - c = 99 & & &
\end{array}$$

EXERCISE CLXXXIV.

Compute the radius of the inscribed circle for the triangles:

	a	b	c
1.	273	425	628
2.	175	527	600
3.	217	404	495
4.	255	407	596
5.	277	304	315

Roots of Proper Fractions.

401. In finding roots of numbers less than 1, where 10 has been added to the logarithm, the resulting logarithm will have an excess of some fraction of 10; thus

$$\log \sqrt[3]{.003987} = \frac{7.6006 \text{ X}}{3}$$

would be too large by $3\frac{1}{3}$, and the subtraction of this excess would be a bother. So before dividing by 3 the excess is made 3×10, and the result turns out to be just 10 too large.

Thus

$$\log .003987 = 27.6006 \text{ XXX}$$

$$\log \sqrt[3]{.003987} = 9.2002 \text{ X}$$

$$\sqrt[3]{.003987} = .1586$$

EXERCISE CLXXXV.

Find the value of:

1. $\sqrt[11]{\dfrac{8.732}{.00658}}$.

2. $\sqrt{\dfrac{31.56 \sqrt[3]{.00937}}{708 \sqrt[5]{.000532}}}$.

3. $\left[\dfrac{83.7}{38700000}\right]^{\frac{4}{5}}$.

4. $\sqrt{\log 3.003}$.

5. $\dfrac{\log 5.38}{\log 3.58} (\log 8.53)$.

6. $(7 + \text{colog } 365)^{\frac{3}{5}}$.

7. $\sqrt[7]{10 + \log .000873}$.

8. $\sqrt{- \log .07}$

9. Find the value of $\dfrac{\log N}{\log b}$ when $N = 15.75$, $b = 21$.

10. If $x \log b = \log N$, find N when $x = 7.832$ and $b = 3.142$.

Evaluate the following formulæ:

11. $\sqrt{s(s - b)(s - c)(s - a)}$ $a = 19$; $b = 23.1$; $c = 37.7$.

12. $\sqrt{\dfrac{(s - b)(s - c)}{s(s - a)}}$ $a = 13$; $b = 18\frac{1}{2}$; $c = 23.1$.

13. $\sqrt[n]{\dfrac{p}{P}}$ $p = .5$; $P = 786$; $n = 50$.

14. $\sqrt[3]{\dfrac{3V}{4\pi}}$ $V = 1728$; $\pi = 3.1416$.

15. $\sqrt[5]{\dfrac{a}{b^2 \sqrt{c}}}$ $a = \log 3$; $b = \log 4$; $c = \log 5$.

16. Find two approximate values that will satisfy the equation $3x^{10} + 8x^5 = 3$.

17. Insert five geometrical means between 8 and 128.

18. Find the value of $\sqrt[9]{9!}$

19. If the first term of a G. P. is 101, and the eleventh term is 800, what is the twelfth term?

20. Solve $(1 - x)^{10} = \dfrac{17!}{6!\,11!}$.

Notation by Powers of Ten.

402. Where a numerical quantity is very large (such as the diameter of the earth's orbit expressed in feet, or the distance light would travel in a year, expressed in miles) or when it is very small (as the length of a wave of light expressed in any ordinary unit) it is customary and very convenient to express the number in question as a multiple of some power of ten.

The distance light would travel in a year is an astronomical standard of measure; thus the Dog-star is eight light-years away, and the pole-star is forty-seven.

Model J.—To find the length of a light-year we must multiply 186,300, the number of miles light travels in a second, by the number of seconds in a year, $3600 \times 24 \times 365.25$

$$\begin{aligned}\log 365.25 &= 2.5626\\ \log 24 &= 1.3802\\ \log 3600 &= 3.5563\\ \log 186300 &= 5.2702\\ \hline &\ 12.7663\end{aligned}$$

The number of miles in a light-year would therefore be 5.839×10^{12}, or 5839×10^9.

EXERCISE CLXXXVI.

1. Find the ratio of 374.8×10^{-8} to seven one-millionths.

2. Find the 200th power of 2.

3. Find the reciprocal of $\left(\frac{28!}{8^{28}}\right)^2$.

4. Find the ratio of $1''$ to $\frac{180°}{\pi}$.

5. Find the 75th term in the expansion of $\left(1 - \frac{\sqrt{3}}{2}\right)^{100}$.

6. A cubic foot of water weighs $62\frac{1}{2}$ pounds, and there are 2000 pounds in a ton; there are 5280 feet in a mile, and 3956.6 miles in the radius of the earth. The volume of the earth is given by the formula $\frac{4}{3}\pi R^3$, and it weighs 5.6 times as much as the same volume of water. Find the weight of the earth in tons.

Other Logarithmic Systems.

403. A table of logarithms can be constructed with any number, commensurable or incommensurable, as a base. The system which has for a base the famous incommensurable number

$$e = 2.718281828\ldots$$

called the **Napierian system**, is of great importance in the higher mathematics.

404. The logarithm of a number to a base other than 10 may be denoted by writing the base as a subscript. Thus, $\log_e x$; $\log_a (x+y)$.

$$\log_2 32 = 5; \text{ because } 2^5 = 32.$$

$$\log_9 \left(\tfrac{1}{3}\right) = -\tfrac{1}{2}; \text{ because } 9^{-\frac{1}{2}} = \tfrac{1}{3}.$$

405. The **base** (b), the **number** (N), and the **logarithm** (x) are connected by the equation (which is a DEFINITION)

$$b^x = N.$$

Model K.—What is the base of a system of logarithms in which the logarithm of 16 is $-1\frac{1}{3}$?

$$b^{-\frac{4}{3}} = 16$$
$$b^{-\frac{1}{3}} = 2$$
$$b^{\frac{1}{3}} = \frac{1}{2}$$
$$b = \frac{1}{8}$$

Model L.—If the logarithm of $\frac{1}{8}$ is 2, what is the logarithm of 16 in the same system?

$$b^2 = \frac{1}{8} = 2^{-3}$$
$$b^x = 16 = 2^4$$
$$b = 2^{-\frac{3}{2}}$$
$$b^x = 2^{-\frac{3x}{2}}$$
$$2^{-\frac{3x}{2}} = 2^4$$
$$x = -\frac{8}{3}. \quad Ans.$$

Since the logarithm of a number in any system is easily calculated from the logarithm of that number in the ordinary system, there is no need of extensive tables for bases other than 10.

EXERCISE CLVXXXII.

1. If the logarithm of 4 is $\frac{2}{3}$, what is the base of the system?

2. If $\log_{16}(x + 5) = .75$, what is the value of x?

3. What is the logarithm of 10 in a system whose base is .01?

4. When $\log_b 81 = -\frac{4}{5}$, what is the value of b?

5. When $\log_{2.25} x = -1.5$, what is the value of x?

6. Find $\log_{25}(.008)$.

7. When p^a is base, what is the log of $p^{(a^2)}$?

8. When p^a is base, what is the log of p^b?

9. When 4 is base, what is the log of 2? of 8? of 4? of 1? of $\frac{1}{32}$? of .03125? of 0?

10. When $\left(\frac{x}{y}\right)^{\frac{a}{b}}$ is base, what is the log of $\left(\frac{y}{x}\right)^{\frac{a}{b}}$? of $\left(\frac{x}{y}\right)^{a}$? of $\left(\frac{x}{y}\right)^{b}$? of $\frac{x}{y}$? of $\left(\frac{x}{y}\right)^{\frac{a}{b}-1}$? of $\left(\frac{x}{y}\right)^{\frac{a^2}{b^2}}$?

11. If log 7 is 3, what is log 343? $\log \sqrt{7}$? $\log \sqrt[3]{49}$?

12. If $b^x = N$ and $b = 10^a$, what is $\log_{10} N$?

406. Model M.—Find $\log_6 7$.

$$6^x = 7$$

$$x \log_{10} 6 = \log_{10} 7$$

$$x = \frac{\log_{10} 7}{\log_{10} 6} = \frac{.8451}{.7782}$$

$$\log .8451 = 9.9270 \text{ X}$$

$$9.8911 \text{ X} \quad \text{colog } .7782 = 0.1089$$

$$\log x = .0359$$

$$x = 1.0863$$

Note that in this example we use the logarithm of a logarithm; thus $9.9270 \text{ X} = \log .8451 = \log \log 7$; and .0359 is not the logarithm we were asked to find, but the logarithm of that logarithm.

EXERCISE CLXXXVIII.

1. Find the log of .01 in a system whose base is 20.

2. What is the base of a system of logarithms in which the logarithm of 20 is 20?

3. Find log 144 to the base $2\sqrt{3}$; to the base 6.

4. For the system whose base is 2, write successively the characteristics of the logarithms for all the natural numbers from 1 to 10.

5. In the system where $\log 2 = 3$, what is $\log 3$?

6. Given $\log 2 = a$; $\log 3 = b$; find $\log 288\sqrt{3}$.

7. Solve the equation $16^{2x} = 25$.

8. Solve the equation $2^x = \frac{57}{90}$.

9. Solve the equation $8^{1+x} = (20\sqrt{17})^3$.

10. Solve the equation $11^x = 2 \log 22$.

11. Insert six geometrical means between 10 and 1000.

12. In G. P. find a formula for n when a, r, and l are given.

13. Find the sum of six terms in G. P., of which the first is 3 and the last 701.

14. The Volunteer Aid Association, desiring to equip a hospital ship, starts an "endless chain" hoping to get \$30,000. (This plan consists in sending out four letters, each numbered "1," and each asking the receiver to do three things: 1°, to make 4 copies, each numbered one more than the copy received; 2°, to mail them to friends; 3°, to mail 10 cents to the Association. The persons receiving letters numbered 20 are to make no copies.) Soon the number of replies becomes owerwhelmingly large, and notices have to be published to stop further contributions. What should have been the last letter-number, to cover the necessary contribution? What would have been the amount received? [Assume that no letter fails.]

15. A sum of money invested in a certain business is expected to double itself in 10 years. Excluding fractional rates of interest, what is the lowest rate that will accomplish this result, assuming interest to be compounded annually?

TABLE OF COMMON LOGARITHMS.

Except in the zero-column only the last three figures of every logarithm are given.

The first figure of each logarithm is the same as the first figure in the same line of the zero-column, except where an asterisk appears.

* *Where the last three figures of the logarithm are preceded by an asterisk, the first figure is the same as the first figure in the* NEXT *line of the zero-column.*

The computer will have to find for himself the value of D between successive logarithms; but in the space to the right of the heavy vertical line are given the values of the fractions .1D, .2D, .3D, .4D, and .5D corresponding to each value of D. This is the table of proportional parts.

N	0	1	2	3	4	5	6	7	8	9
10	0000	043	086	128	170	212	253	294	334	374
11	0414	453	492	531	569	607	645	682	719	755
12	0792	828	864	899	934	969	*004	*038	*072	*106
13	1139	173	206	239	271	303	335	367	399	430
14	1461	492	523	553	584	614	644	673	703	732
15	1761	790	818	847	875	903	931	959	987	*014
16	2041	068	095	122	148	175	201	227	253	279
17	2304	330	355	380	405	430	455	480	504	529
18	2553	577	601	625	648	672	695	718	742	765
19	2788	810	833	856	878	900	923	945	967	989
20	3010	032	054	075	096	118	139	160	181	201
21	3222	243	263	284	304	324	345	365	385	404
22	3424	444	464	483	502	522	541	560	579	598
23	3617	636	655	674	692	711	729	747	766	784
24	3802	820	838	856	874	892	909	927	945	962
25	3979	997	*014	*031	*048	*065	*082	*099	*116	*133
26	4150	166	183	200	216	232	249	265	281	298
27	4314	330	346	362	378	393	409	425	440	456
28	4472	487	502	518	533	548	564	579	594	609
29	4624	639	654	669	683	698	713	728	742	757
30	4771	786	800	814	829	843	857	871	886	900
31	4914	928	942	955	969	983	997	*011	*024	*038
32	5051	065	079	092	105	119	132	145	159	172
33	5185	198	211	224	237	250	263	276	289	302
34	5315	328	340	353	366	378	391	403	416	428
35	5441	453	465	478	490	502	514	527	539	551
36	5563	575	587	599	611	623	635	647	658	670
37	5682	694	705	717	729	740	752	763	775	786
38	5798	809	821	832	843	855	866	877	888	899
39	5911	922	933	944	955	966	977	988	999	*010
40	6021	031	042	053	064	075	085	096	107	117
41	6128	138	149	160	170	180	191	201	212	222
42	6232	243	253	263	274	284	294	304	314	325
43	6335	345	355	365	375	385	395	405	415	425
44	6435	444	454	464	474	484	493	503	513	522
45	6532	542	551	561	571	580	590	599	609	618
46	6628	637	646	656	665	675	684	693	702	712
47	6721	730	739	749	758	767	776	785	794	803
48	6812	821	830	839	848	857	866	875	884	893
49	6902	911	920	928	937	946	955	964	972	981
50	6990	998	*007	*016	*024	*033	*042	*050	*059	*067
51	7076	084	093	101	110	118	126	135	143	152
52	7160	168	177	185	193	202	210	218	226	235
53	7243	251	259	267	275	284	292	300	308	316
54	7324	332	340	348	356	364	372	380	388	396

D	1	2	3	4	5
43	4	9	13	17	22
42	4	8	13	17	21
41	4	8	12	16	20
40	4	8	12	16	20
39	4	8	12	16	20
38	4	8	11	15	19
37	4	7	11	15	18
36	4	7	11	14	18
35	4	7	10	14	18
34	3	7	10	14	17
33	3	7	10	13	16
32	3	6	10	13	16
31	3	6	9	12	16
30	3	6	9	12	15
29	3	6	9	12	14
28	3	6	8	11	14
27	3	5	8	11	14
26	3	5	8	10	13
25	2	5	8	10	12
24	2	5	7	10	12
23	2	5	7	9	12
22	2	4	7	9	11
21	2	4	6	8	10
20	2	4	6	8	10
19	2	4	6	8	10
18	2	4	5	7	9
17	2	3	5	7	8
16	2	3	5	6	8
15	2	3	4	6	8
14	1	3	4	6	7
13	1	3	4	5	6
12	1	2	4	5	6
11	1	2	3	4	6
10	1	2	3	4	5
9	1	2	3	4	4
8	1	2	2	3	4

N	0	1	2	3	4	5	6	7	8	9	D	1	2	3	4	5
55	7404	412	419	427	435	443	451	459	466	474	8	1	2	2	3	4
56	7482	490	497	505	513	520	528	536	543	551						
57	7559	566	574	582	589	597	604	612	619	627						
58	7634	642	649	657	664	672	679	686	694	701						
59	7709	716	723	731	738	745	752	760	767	774						
60	7782	789	796	803	810	818	825	832	839	846	7	1	1	2	3	4
61	7853	860	868	875	882	889	896	903	910	917						
62	7924	931	938	945	952	959	966	973	980	987						
63	7993	*000	*007	*014	*021	*028	*035	*041	*048	*055						
64	8062	069	075	082	089	096	102	109	116	122						
65	8129	136	142	149	156	162	169	176	182	189	6	1	1	2	2	3
66	8195	202	209	215	222	228	235	241	248	254						
67	8261	267	274	280	287	293	299	306	312	319						
68	8325	331	338	344	351	357	363	370	376	382						
69	8388	395	401	407	414	420	426	432	439	445						
70	8451	457	463	470	476	482	488	494	500	506						
71	8513	519	525	531	537	543	549	555	561	567						
72	8573	579	585	591	597	603	609	615	621	627						
73	8633	639	645	651	657	663	669	675	681	686						
74	8692	698	704	710	716	722	727	733	739	745						
75	8751	756	762	768	774	779	785	791	797	802						
76	8808	814	820	825	831	837	842	848	854	859						
77	8865	871	876	882	887	893	899	904	910	915						
78	8921	927	932	938	943	949	954	960	965	971						
79	8976	982	987	993	998	*004	*009	*015	*020	*025						
80	9031	036	042	047	053	058	063	069	074	079	5	0	1	2	2	2
81	9085	090	096	101	106	112	117	122	128	133						
82	9138	143	149	154	159	165	170	175	180	186						
83	9191	196	201	206	212	217	222	227	232	238						
84	9243	248	253	258	263	269	274	279	284	289						
85	9294	299	304	309	315	320	325	330	335	340						
86	9345	350	355	360	365	370	375	380	385	390						
87	9395	400	405	410	415	420	425	430	435	440						
88	9445	450	455	460	465	469	474	479	484	489						
89	9494	499	504	509	513	518	523	528	533	538						
90	9542	547	552	557	562	566	571	576	581	586	4	0	1	1	2	2
91	9590	595	600	605	609	614	619	624	628	633						
92	9638	643	647	652	657	661	666	671	675	680						
93	9685	689	694	699	703	708	713	717	722	727						
94	9731	736	741	745	750	754	759	763	768	773						
95	9777	782	786	791	795	800	805	809	814	818						
96	9823	827	832	836	841	845	850	854	859	863						
97	9868	872	877	881	886	890	894	899	903	908						
98	9912	917	921	926	930	934	939	943	948	952						
99	9956	961	965	969	974	978	983	987	991	996						

CHAPTER XVI.

SUPPLEMENTARY PROBLEMS FOR PRACTICE AND REVIEW.

407. The problems in this chapter are selected, for the most part, from examination papers, set by colleges or institutions of a similar grade. The arrangement of them is purposely left somewhat irregular, so that the attitude of the pupil towards his work may be more natural than when, in doing a set of carefully classified exercises, he knows beforehand the kind of difficulty he is about to meet. This is especially noticeable in the "concrete problems" which close the collection.

In the absence of other directions, where an algebraic expression is given, simplify it by performing the operations indicated: where an equation or a set of equations are given, solve them for what seem to be the unknown letters.

Factor the following expressions:

1. $x^2 - 8x - 84$. **2.** $(2x+3)^2 - (x-3)^2$.

3. $x^2 - x - 2$. **4.** $324a^4b^2 - 64b^6$. **5.** $x^2 + 12x - 85$.

6. $3(x+y)^2 - 2(x^2-y^2) - x(x+y)$. **7.** $x^2 - 2xy - xz + 2yz$.

8. $x^2 - 2xy - z^2 + y^2$. **9.** $y^3 + 4y^2 + 4y$.

10. $x^2y^2 - x^2 - y^2 + 1$. **11.** $a^3 + a^2 - a - 1$.

12. $12x^2 - 5x - 2$. **13.** $12x^2 - x - 1$. **14.** $(x+1)^4 - 1$.

15. $18x^3 - 18x^2 + 4x$. **16.** $2x^4 - 3x^2 - 14$. **17.** $x^2 - x - 306$.

18. $8x^2 - 39x + 46$. **19.** $(x^2 - x)^3 - 8$. **20.** $x^{16} - a^{16}$.

21. $8cx - 12cy + 2ax - 3ay$. **22.** $a^4 + 4b^2 + 4a^2b^2 - c^8$.

23. $2am - b^2 + m^2 + 2bn + a^2 - n^2$. **24.** $3ax - bx - 3ay + by$.

25. $x^2 - 20xy - 96y^2$. **26.** $2x^2 + 5xy - 12y^2$.

27. $a^2 - ab - b - 1$. **28.** $a^{12} - b^{12}$. **29.** $a^8 + a^4b^4 - 2b^8$.

30. $(x - 1)(x - 2)(x - 3) + (x - 1)(x - 2) - x - 1$.

31. $a^2y^2 + a^2x^2 - b^2x^2 - b^2y^2$. **32.** $6m^2 - 38m - 28$.

33. $a^2 + x^2 + 2ax - y^2$. **34.** $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

35. $4x^2 - 6yz - (9y^2 + z^2)$. **36.** $9y^2 + 4xz - (4x^2 + z^2)$.

37. $5x^4 - 15x^3 - 90x^2$. **38.** $a^2 + x^2 - (y^2 + z^2) - 2(yz - ax)$.

39. $y^9 - b^9$. **40.** $x^{2m} + \frac{1}{2}x^m + \frac{1}{16}$. **41.** $9x^4 - 37x^2 + 4$.

42. $5x^4 - 15x^3 - 90x^2$. **43.** $(a + b)^2x^2 - (a^2 - b^2)x - ab$.

44. $x^3 - x^2 - x + 1$. **45.** $(a^2 - b^2 - c^2)^2 - 4b^2c^2$.

46. $x^2 + (a + b + c)x + ab + ac$.

47. Having given m, the difference of the squares of any two consecutive numbers, find the numbers. Verify by substituting figures for letters.

48. Give three different forms of polynomials that are factorable, and illustrate the method of factoring by an example.

49. Find and factor the expression which added to the following will reduce it to zero:

$$(b - a)(a + b - c) + (c - b)(b + c - a)$$

and check the result by substituting $a = \frac{1}{2}$; $b = -\frac{2}{3}$; $c = \frac{3}{4}$.

50. Subtract $3x^3 - 7x + 1$ from $2x^2 - 5x - 3$, then subtract the difference from zero, and add this last result to $2x^2 - 2x^3 - 4$.

51. $x - [x - 2x - (x - 3x) - (x - 4x)]$.

52. $(3x - 1)^3 - (2x^2 - 1)(3x + 1)$
$$- 3x[(1 - 3x)^2 - (2x^2 - 1) - 2(2x - 1)].$$

53. If $x = -1\frac{1}{3}$, find the value of

$$\frac{(1-x)(1+3x)}{1-3x} - \frac{(1+x)(1-3x)}{1+3x}.$$

54. Find fcur factors of $3(6x^2+5x)^2 - 10(6x^2+5x) - 8$.

55. $2x - \Big[a - (3b - 5c) + [x - (4a - \overline{b + x - c}) + 3c] - [2b - \overline{3a + 3c}]\Big]$.

56. $\dfrac{a(b-c)(c-a)(a-b) - b(c-b)(a-c)(b-a)}{(a+b)(c-a)(b-c)}$.

57. What value of a will make $6x^4 - 2x^3 + 2ax^2 + 2x + a$ an exact multiple of $x^2 - x + 1$?

58. Assign a value to b which will make $\dfrac{b-4}{b-5}$ equal to $\dfrac{b+5}{b-4}$.

59. Subtract the sum of the squares of $ax + by$ and $ay - bx$ from the product of $a^2 + y^2$ and $b^2 + x^2$.

60. Supposing that $a + b - c = 0$, simplify $\dfrac{a^2 + bc}{b^2 + ac}$.

61. What fraction must be added to $\dfrac{4x}{(x-1)^2(x+1)} - \dfrac{x+1}{(x-1)^2}$ to make its value 1?

62. $3(x+z) - (6a - z) - 2[x - (2a + z) - (a - 3z)]$.

63. Express $\dfrac{x}{x^2-9} - \dfrac{1}{x-3}$ as a single fraction, with $(3-x)(3+x)^2$ for denominator.

64. Find an expression such that when divided by $1 + 2a + 3a^2 + 4a^3$ it will give $1 - 2a + 3a^2 - 4a^3$ for a quotient, and $5x^4 - 6x^5$ for a remainder.

65. $\dfrac{a^3 - b^3}{a^2 - 5ab + 6b^2} \div \dfrac{a-b}{a^2 + 3ab - 10b^2}$.

66. Find the value of x from the proportion

$$\frac{1}{3a^3 - 4a^2 - 4a + 5} : \frac{1}{6a^3 + 7a^2 - 13a - 15} = (a+1) : x.$$

67. Reduce to a single fraction in its lowest terms:

$$\frac{1}{x^3 - x} - \left(\frac{1}{x^3 - 1} - \frac{1}{x^3 + 1}\right).$$

68. Find the square root of

$$4x^6 - 20x^5 + 21x^4 + 22x^3 - 29x^2 - 6x + 9.$$

69. Reduce

$$\frac{5}{9x^4 - x^2 + 10x - 25} - \frac{1}{6x^4 - 2x^3 + 7x^2 + x - 5}$$

to its lowest terms as a single fraction.

70. Find the following expressions in their lowest terms:

$$\frac{2x + 1}{12x^3 - 10x^2 - 6x + 4} - \frac{x + 1}{12x^3 + 8x^2 - 3x - 2}$$

and

$$\frac{2x + 1}{12x^3 - 10x^2 - 6x + 4} \div \frac{x + 1}{12x^3 + 8x^2 - 3x - 2}.$$

71. Reduce to their lowest common denominator

$$\frac{1}{12x^3 - 2x^2 - 20x - 6} \text{ and } \frac{1}{4x^3 - 6x^2 - 4x + 6};$$

and find, and reduce to their lowest terms, the *difference* and the *quotient* of these two fractions.

72. If $\dfrac{x}{y + z} = a$; $\dfrac{y}{x + z} = b$; $\dfrac{z}{x + y} = c$; find the value of $\dfrac{a}{1 + a} + \dfrac{b}{1 + b} + \dfrac{c}{1 + c}$.

73. Find the value of $\dfrac{x^5 - x^4y + x^3y^3}{y^5 - y^4x + y^3x^3}$ when $x = 3y$.

74. Find the value of $(x+y)(x-y)(x^4+y^4)$ when the sum of the squares of x and y is 3, and the difference of their eighth powers is 21, y being greater than x.

75. If $a=0$, $b=1$, $c=\frac{1}{2}$, $d=-1$, find the value of $\frac{a^2-b^2}{c-d}+\frac{b^2-c^2}{d-a}+\frac{c^2-d^2}{a-b}$.

76. $\left(x^{2m}-x^m-x^{\frac{1}{m}}-x^{\frac{2}{m}}\right)\left(x^m-x^{\frac{1}{m}}+1\right)$.

77. What value of x will make $x^5+7x^3-49x^2+8x+2585$ an exact multiple of x^2-7x+1?

78. Find the value of $(a-b)(a+b)-(a+b)^2$ when $3a+2c=45$ and $3c+2a=15$.

79. Find two expressions whose product is $12x^3-40x^2+39x-9$ and whose quotient is $3x-1$.

80. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$ when $x=\frac{4ab}{a+b}$.

81. Find the value of

$$x(y+z)+y[x-(y+z)]-z[y-x(z-x)]$$

when $x=3$, $y=2$, $z=1$.

82. Substitute $1-x^2$ for y in $\dfrac{\dfrac{2y+4x^2}{y^2}}{1+\dfrac{4x^2}{y^2}}$ and simplify.

83. Find the value of $\frac{x^2-y^2}{x^2+y^2}$ when $x=\frac{a+b}{a-b}$ and $y=\frac{a-b}{a+b}$.

84. Find the value of $\frac{x-a}{b}-\frac{x-b}{a}$ when $x=\frac{a^2}{a-b}$.

85. Find the value of $\frac{10x-3y}{2x-y}$ when $\frac{x+6y}{7x-2y}=8$.

86. Show that the difference of the cubes of any two consecutive numbers is one more than three times their product.

87. What values has the expression $\dfrac{7x^2 - 4y^2}{x^2 + 2xy}$ when $\dfrac{x(x - 2y)}{3y} = x - 2y$?

88. What value must be assigned to $\dfrac{x}{y}$ in order that $2x^2 - xy = 2xy + 2y^2$?

89. A sum of \$100 is put at compound interest at 4 per cent per annum for x years; find a formula for the amount.

90. What value must y have in order that when $x = 1$ the equation $3x + 2y - 1 = 2x + 5y - 18$ will be true?

91. What value must be given to x in order that when $a = \frac{2}{3}$ the following expressions may be equal?

$$a\left(x - \frac{1}{a}\right) + \frac{1}{2a}\left(x - 2a\right);\ \left(a + \frac{1}{6}\right)\left(x + \frac{6}{5}\right) + \left(a + \frac{1}{4}\right)\left(x - \frac{12}{11}\right) + 5.$$

92. Find the sum of $\dfrac{a}{bc}$, $\dfrac{b}{ca}$, and $\dfrac{c}{ab}$; subtract $2\left(\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{1}{c}\right)$ from the result, and simplify the remainder.

93. Assuming that $x^6 - 6x^5 + 3x^4 + 28x^3 - 9x^2 - 54x - 27$ is a perfect cube, find what values of x will cause it to vanish.

94. Find the difference of the squares of the highest and lowest of any three consecutive even numbers, and express the result as a theorem. Is this theorem true of odd numbers?

95. If the quadratic $7x^2 - 2x + (2 - c)$ is a perfect square, what is the value of c?

96. Prove that when a number of two digits is equal to four times the sum of its digits, one digit is double the other.

97. Two men, working separately, can do a piece of work in x days and y days, respectively; find an expression for the time in which both can do it, working together.

98. A is 20 years old, and B is -2 years older; what is the age of B?

99. What are the values of x which satisfy the equation $x^2 = 3x$?

100. Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$ and factor the quotient.

101. Find the H. C. F. of $6x^4 - 2x^3 + 9x^2 + 9x - 4$ and $9x^4 + 80x^2 - 9$; what value of x will make both these expressions vanish?

102. Find the H. C. F. of $3x^5-5x^3+2$ and $2x^5-5x^2+3$.

103. Find the L. C. M. of $13ab^2(x^3 - 3a^2x + 2a^3)$; $65a^3b(x^2 + ax - 2a^2)$; and $25b^3(x^2 - a^2)^2$.

104. Find the G. C. M. of $x^3 - 93x - 308$ and $x^3 - 21x^2+131x-231$; and the L. C. M. of $12x^2y(x^3+y^3)$; $18xy^2(x^3 - y^3)$; $21x^2y^2(x^4 + x^2y^2 + y^4)$.

105. Find the H. C. D. of $x^4 - 2x^3 + 4x^2 - 6x + 3$ and $x^4 - 2x^3 - 2x^2 + 6x - 3$; and the L. C. M. of $3a^2x^3$; $5ax^4$; $35a^4$; and $15a^2x^3$.

106. Find the H. C. D. of $x^3 - 40x + 63$ and $x^4 - 7x^3 + 63x - 81$; and the L. C. M. of $7a^2x(a-x)$; $21ax(a^2 - x^2)$; $12ax^2(a+x)$.

107. Find the G. C. M. of $15a^4 + 10a^3b + 4a^2b^2 + 6ab^3 - 3b^4$ and $6a^3 + 19a^2b + 8ab^2 - 5b^3$; and the L. C. M. of $(s-t)^2(a-b)^3$; $(t+s)^2(a^2-b^2)^4$; $(a+b)^4(b+c)$.

108. Find the G. C. M. of x^5+11x^3-54 and $x^5+11x+12$; and the L. C. M. of $25(a^3+b^3)(a^4-b^4)$; $30ab(a^2+b^2)$; $45b(a^3-b^3)$.

109. If $x^2 + 9x + a$ is exactly divisible by $x + 5$, what is the value of a?

110. L. C. M. of $9x^3 - x - 2$ and $4 + 7x + 10x^2 - 3x^3$.

111. Prove that the G. C. D. of $2 + x - x^2 - 2x^3$ and $x^5 - x^3 - 2x^2 + 2x$ is the square root of a factor of the second expression.

112. Find the L. C. M. of $x^2 + x^5$ and $x^{17} + x^8$.

113. Supposing that $x^4 + 4x^3 - 2x^2 - 12x + 9$ is a perfect square, find its prime factors.

114. Supposing that $4x^4 - 12x^3 + 17x^2 - 12x + 4$ is a perfect square, find what value of x will make it vanish.

115. Write two expressions of the fourth degree which have $3x^2 - 2x + 5$ for their H. C. F.

116. Prove that $y^2 - 4y$ is 4 less than a common factor of $y^7 - 6y^6 + 13y^5 - 12y^4 + 4y^3$ and $y^6 - 5y^5 + 8y^4 - 4y^3$; and find their H. C. F.

117. G. C. D. and L. C. M. of $6x^4 - 5x^3 - 10x^2 + 3x - 10$ and $4x^3 - 4x^2 - 9x + 5$.

118. G. C. D. and L. C. M. of $12x^2 - 29x + 14$ and $18x^2 + 3x - 10$.

119. G. C. M. and L. C. M. of $6x^3 + 7x^2 - 5x$ and $15x^4 + 31x^3 + 10x^2$.

120. G. C. M. of $2a^3 - 15a + 14$ and $2a^4 - 30a^2 + 56a - 24$.

121. G. C. F. of $x^3 + 5x^2 + 6x$ and $3x^3 + 7x^2 + 3x + 2$.

122. What value of m will make $x^2 - (3m - 1)x + 2m$ exactly divisible by $x - 1$?

123. G. C. D. and L. C. M. of $x^7y^5 - x^5y^7$ and $x^4 + xy^3$.

124. G. C. M. of $8x^4 - 12x^3 + 2x^2 + 3x - 1$ and $6x^3 - 7x^2 + 1$.

125. The expression $x^4 + 4x^3 - 2x^2 - 12x + 9$ has two identically equal factors; what values of x will make it vanish?

126. G. C. M. and L. C. M. of $2a^3 + 5a^2b - 5ab^2 + b^3$ and $2a^3 - 7a^2b + 5ab^2 - b^3$.

127. Find two values of x that will satisfy both of the following equations: $4x^4 - 1 = 9x^2 - 6x$ and $6x^3 + 1 = 7x^2$.

128. G. C. D. and L. C. M. of $2x^4 - 7x^3 + 8x^2 - 10x + 3$ and $3x^4 - 10x^3 + 8x^2 - 7x + 2$.

129. H. C. F. of $x^3 - 3x + 2$ and $x^3 + x^2 - 5x + 3$.

130. Reduce to lowest terms

$$\frac{6x^5 - 2x^4 - 11x^3 + 5x^2 - 10x}{9x^5 + 3x^4 - 11x^3 + 9x^2 - 10x}.$$

131. Given the three expressions $2x^4+x^3-8x^2-x+\ ;$ $4x^4+12x^3-x^2-27x-18$; $4x^4+4x^3-17x^2-9x+18$; find the G. C. D. and the L. C. M. of the first two, also of the whole group of three.

132. Reduce to lowest terms

$$\frac{6x^4-13x^3+3x^2+2x}{6x^4-9x^3+15x^2-27x-9}.$$

133. Subtract from the sum of $\frac{a+b}{a-b}$ and $-\frac{a-b}{a+b}$ the product of $-\frac{b-a}{b+a}$ and $-\frac{b+a}{b-a}$; and divide the remainder by $-\left(\frac{a}{b}-\frac{b}{a}\right)$.

134. Express as a theorem the general law observable in the following equations:

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \end{aligned}$$

and so on.

135. $\left(\frac{1+x}{1-x}-\frac{1-x}{1+x}\right)\div\left(\frac{1+x}{1-x}-1\right)$.

136. $\frac{a^2b^2}{c}\div\left(\frac{a^2c^2}{b}\div\frac{b^2c^2}{a}\right)$. **137.** $\frac{x^4-10x^2+9}{x^4-4x^2+3}$.

138. $$\frac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}}\times\frac{x^2-y^2}{x^3+y^3}.$$

139. $$\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(a-c)(b-c)+(b-a)(c-a)+(c-b)(a-b)}.$$

140. Multiply $(a-b)^2-ab$ by $(a+b)^2+ab$.

141. Divide $(x-y)^4+(x^2-y^2)^2+(x+y)^4$ by
$(x-y)^2+x^2-y^2+(x+y)^2$.

142. $\dfrac{a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3}{bc+ca+ab}$.

143. $\dfrac{1}{a+\dfrac{1}{a+2}}\times\dfrac{1}{a+\dfrac{1}{a-2}}\times\dfrac{a^3+\dfrac{1}{a}-2a}{a^2-4}$.

144. $\dfrac{3(a^2+a-2)}{a^2-a-2}-\dfrac{3(a^2-a-2)}{a^2+a-2}-\dfrac{8a}{a^2-4}$.

145. Divide $(a^3-2a^2b+3ab^2-2b^3)(a^3+2a^2b+3ab^2+2b^3)$ by $a^2+ab+2b^2$.

146. $\left\{2+\dfrac{2+\dfrac{2+x}{2-3x}}{2-3\dfrac{2+x}{2-3x}}\right\}\div\left\{2-3\dfrac{2+\dfrac{2+x}{2-3x}}{2-3\dfrac{2+x}{2-3x}}\right\}$

147. $\dfrac{a^4-b^4}{a^2b^2}\left(\dfrac{a^2}{a^2-b^2}-1+\dfrac{b^2}{a^2+b^2}\right)$.

148. Substitute $A=a^2-bc$; $B=b^2-ac$; $C=c^2-ab$: in $\dfrac{A}{ac}+\dfrac{B}{ab}+\dfrac{C}{bc}$.

149. $\dfrac{(p^2+q^2)(x+y)^2+2(px-qy)(qx-py)}{(p^2-q^2)(x^2+y^2)}$.

150. $\dfrac{(x^3+8)[x(x-2)-4]}{[x^2-2x+4][x(x^2-8)-8]}$.

151. Express as a theorem the fact implied in the following equations: $9^2-7^2=4\times 8$; $5^2-3^2=4\times 4$; $93^2-91^2=4\times 92$. Prove it, and then express the theorem in the most general way.

152. Reduce $\dfrac{x^6-9x^3+8}{x^4-5x^2+4}$; and find its value when $x=1$; also when $x=2$.

153. $\sqrt{(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2}$.

154. $\dfrac{\dfrac{x}{x-1} - \dfrac{x+1}{x}}{\dfrac{x}{x+1} - \dfrac{x-1}{x}}$.

155. $x + 1 - \dfrac{x}{x + 2 - \dfrac{x+1}{x + \dfrac{1}{x+2}}}$.

156. $\left[\dfrac{1}{b} + \dfrac{10}{3a-b} - \dfrac{6}{a-3b}\right] \div \left[\dfrac{3(a+b)}{3a-b} - \dfrac{a-b}{a-b-\dfrac{8b^2}{a+b}}\right]$.

157. $x^5 - 4bx^4 - \dfrac{1}{6}\left[12ax - 4\left\{3bx^4 - 9\left(\dfrac{cx}{2} - bx^5\right) - \dfrac{3}{2}ax^4\right\}\right]$.

158. $\left(1 + \dfrac{a}{1-a}\right) \times \dfrac{1-a^2}{1+b} \times \dfrac{1-b^2}{a+a^2}$.

159. $\dfrac{2x}{3} \div \left[\dfrac{1}{x} - \dfrac{2x^3 + 11x^2 - 43x - 24}{14x^3 - 31x^2 - 31x - 6}\right]$.

160. $\dfrac{a^2c + abc + b^2c}{a^4 + a^2b^2 + b^4}$.

161. $\dfrac{\dfrac{1-x^2}{1+y}\left(\dfrac{x}{1+x} - 1\right)}{1 - \left(\dfrac{1}{1-y} - \dfrac{x^2+y^2-x+y}{1-y^2}\right)}$.

162. $\dfrac{\dfrac{1-x^3}{1+x^3} - \dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2} + \dfrac{1+x}{1-x}}$.

163. $\dfrac{\dfrac{a^2}{b^3} + \dfrac{1}{a}}{\dfrac{a}{b} - \dfrac{1}{b} + \dfrac{1}{a}}$.

164. $\dfrac{1 - \dfrac{2 + \dfrac{1}{x}}{2 - \dfrac{1}{x}}}{1 - \dfrac{x + \dfrac{1}{2}}{x - \dfrac{1}{2}}}$.

165. $\left[\dfrac{1}{a + \dfrac{1}{b + \dfrac{1}{c}}} \div \dfrac{1}{a + \dfrac{1}{b}}\right] - \dfrac{1}{b(abc + a + c)}$.

166. Compute the value of the continued fraction

$$\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{5}}}}$$

167. $\dfrac{1}{x(x-a)(x-b)}+\dfrac{1}{a(a-x)(a-b)}+\dfrac{1}{b(b-x)(b-a)}.$

168. Divide $\dfrac{c-b}{c+b}-\dfrac{c^2-b^2}{c^2+b^2}$ by $\dfrac{c+b}{c-b}+\dfrac{c^2+b^2}{c^2-b^2}.$

169. Compute the value of the continued fraction

$$\cfrac{1}{12+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}}$$

170. $\cfrac{1}{x+\cfrac{2}{1-\cfrac{x-2}{2x+1}}}.$

171. $\dfrac{ab}{b^2-4x^2}+\dfrac{cd}{ab+2ax}.$

172. $\left(2a+3b-\dfrac{24ab}{2a+3b}\right)\times\left(2a-3b+\dfrac{24ab}{2a-3b}\right).$

173. $\left[\dfrac{y^2-yz+z^2}{x}+\dfrac{x^2}{y+z}-\dfrac{3}{\frac{1}{y}+\frac{1}{z}}\right]\dfrac{\frac{2}{y}+\frac{2}{z}}{\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}}+(x+y+z)^2.$

174. $\left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)(a^2+2ab-b^2)\div\left(\dfrac{a}{a-b}+\dfrac{b}{a+b}\right).$

175. $\dfrac{a^4+b^4+ab(a^2+b^2)+a^2b^2}{a^5-b^5}\div\dfrac{a^2+b^2+ab}{a^3-b^3}.$

176. $\dfrac{(a-b)^4 - ab(a-b)^2 - 2a^2b^2}{(a-b)(a^3-b^3) + 2a^2b^2}.$

177. $\dfrac{a^6+b^6}{a^6-b^6} \times \dfrac{a-b}{a+b} \div \dfrac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}.$

178. $\dfrac{1}{2(a-1)} - \dfrac{a-5}{a^2-7a+10} + \dfrac{1}{2} \cdot \dfrac{a-6}{a^2-9a+18}.$

179. $\dfrac{\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} - 2}{\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} + 2} \div \dfrac{\dfrac{x}{y}\left(1 - \dfrac{y^2}{x^2}\right)}{\dfrac{(x+y)^2}{xy} - 2}.$

180. $\dfrac{x}{(x+3)(x-1)} + \dfrac{x-1}{x+3)(2-x)} - \dfrac{x-3}{(2-x)(1-x)}.$

181. $\dfrac{x}{3 + \dfrac{x^2}{4 + \dfrac{5}{x^4}}}.$

182. $\dfrac{1}{y}\left(\dfrac{1}{x-y} + \dfrac{1}{x+2y}\right) - \dfrac{3}{x^2+xy-2y^2}.$

183. $\dfrac{y(y-1)(y-2)(y-3)+1}{y^2-3y+1}.$

184. $\dfrac{1 - \dfrac{y}{x} + \dfrac{y^2}{x^2}}{1 + \dfrac{y}{x} + \dfrac{y^2}{x^2}} \times \dfrac{\dfrac{x^3}{y^3} - 1}{\dfrac{x^3}{y^3} + 1} \div \dfrac{\left(\dfrac{1}{x} - \dfrac{1}{y}\right)^2}{\left(\dfrac{1}{x} + \dfrac{1}{y}\right)^2}.$

185. $\dfrac{x-1}{x^2+7x+10} - \dfrac{2x+4}{x^2+4x-5} + \dfrac{x+5}{x^2+x-2}.$

186. $3x - 4y + 2z = 1;\ 4x - 2y + 3z = 9;\ 2x - 3y + 4z = 8.$

187. $x + y - 3z = 8;\ y + z - 3x = -4;\ z + x - 3y = -8.$

188. $4x + 2(3x - y - 1) = 1 + 3(y - 1);$

$$\tfrac{1}{10}(4x + 3y) = \frac{7y}{20} + 1.$$

189. $\frac{5x-2y}{17}-\frac{2x+3y}{3}=x+y-2;\ \frac{3x-4y}{13}=\frac{3x+4y}{5}.$

190. Find the values of a, b, and c in the following equations:

$$\begin{cases} 3a+\frac{3}{2}b=30-c \\ \frac{1}{2}(a+c)-(a-c)=\frac{1}{5}(9-b) \\ \frac{3}{4}(a-c)=3(2b-7) \end{cases}$$

191. $\frac{13}{x+2y+3}=-\frac{3}{4x-5y+6};\ \frac{3}{6x-5y+4}=\frac{19}{3x+2y+1}.$

192. $3x-5y=\frac{2x-y}{7}-1-\frac{5x-2y}{4};\ \frac{x+y}{x-y}=11.$

193. $4(x-2y+4)=2x+3(y-\frac{1}{2});\ 5\left(y+\frac{x}{2}\right)-2(x+2)=11.$

194. $8x-y=2x+2y=6.$

195. $\frac{1}{z}-2y=7x-\frac{3}{z}=y-2x=-1.$

196. $3x-y+2z=3y-z+2x+2=3z-x+2y-5=11.$

197. $x-5=\frac{1}{7}(y-2);\ 4y-3=\frac{1}{3}(x+10).$

198. $4x-6y-4=7x+2y-5=-2x+3y+23.$

199. $\frac{1}{3x}+\frac{1}{5y}=\frac{2}{9};\quad \frac{1}{5x}+\frac{1}{3y}=\frac{1}{4}.$

200. $\frac{7}{\sqrt{x}}+\frac{4}{\sqrt{y}}=4;\quad \frac{1}{\sqrt{x}}+\frac{2}{\sqrt{y}}=1.$

201. $\begin{cases} x+y=\frac{2(a^2+b^2)}{a^2-b^2}. \\ x-y=\frac{4ab}{a^2-b^2}. \end{cases}$

202. $2x+4y+27z=28;\quad 7x-3y-15z=3;$
$9x-10y-33z=4.$

203. $5x-3y+2z=41;\ 2x+y-z=17;\ 5x+4y-2z=36.$

204. Solve the equation $6-\frac{5x+1}{4}=\frac{21-3x}{3}-\frac{2(2x+3)}{9}.$

205. $\dfrac{5x-3}{4}=\dfrac{3x-5}{10}+\dfrac{1}{20}+\dfrac{8(x-1)}{5}.$

206. $\dfrac{x-2}{3x-1}+\dfrac{7}{9}\left(\dfrac{3x-8}{x-3}\right)=2\tfrac{2}{3}.$

207. $1+x^3=5\tfrac{1}{4}+\dfrac{x^3-4\frac{1}{4}}{x}.$

208. $(7+x)(8-x)-1=\dfrac{7x}{3}+17x-x^2.$

209. $\dfrac{23}{10(x-1)}+\dfrac{3}{5}\left(\dfrac{1}{x-1}-\dfrac{1}{3}\right)=\dfrac{x-\frac{1}{2}}{x-1}.$

210. $\dfrac{1}{x+8}+\dfrac{1}{x+4}=\dfrac{1}{x+2}+\dfrac{1}{x+10}.$

211. $\dfrac{5x-12\frac{1}{2}}{9}+\dfrac{3x-2\frac{1}{2}}{5}=\dfrac{6x+7}{11}-\dfrac{2x+3}{8}.$

212. $3x-5+\sqrt{8(2x-3)(x+1)-(7x-5)(x+2)}=0.$

213. $\dfrac{2x+3}{18}+\dfrac{x-\frac{1}{2}}{6}=\tfrac{1}{5}[x+\tfrac{2}{5}(x+1)].$

214. $\dfrac{x^2+1}{x^2-1}+\dfrac{x^2-2}{x^2-x-2}=\dfrac{2x}{x+1}.$

215. $(3x-1)(4x+5)-(2x+3)(5x-2)$
$=(x-2)(2x+1)+12.$

216. $10\left(x+\dfrac{1}{2}\right)-6x\left(\dfrac{1}{x}-\dfrac{1}{3}\right)=23.$

217. $\dfrac{x+3}{x+2}+\dfrac{x-1}{x+1}=\dfrac{2x-3}{x-1}.$

218. $x-\dfrac{x-2}{3}=\dfrac{x+23}{4}-\dfrac{10+x}{5}.$

219. $\dfrac{5x+2}{3}-\left(3-\dfrac{3x-1}{2}\right)=\dfrac{3x+19}{2}-\left(\dfrac{x+1}{6}+3\right).$

220. $\dfrac{7x+9}{4}-\left(x-\dfrac{2x-1}{9}\right)=7.$

221. $\dfrac{1}{x^2-4}+\dfrac{3}{x-2}-\dfrac{2}{x+2}=1.$

222. $3x - 4(2x + 3)^2 + 190 = 0.$ **223.** $x^2 - 4x + 1 = 0.$

224. $\dfrac{2x - 1}{x + 1} = \dfrac{x + 1}{x - 2}.$ **225.** $\sqrt{x^2 + 16x} + 7 = 3x - 5.$

226. $4x^{-\frac{4}{3}} - 5x^{-\frac{2}{3}} + 1 = 0.$

227. $\dfrac{1}{2x - 3} = \dfrac{4}{5x - 6} + \dfrac{7}{8x - 9} - \dfrac{10}{11x - 12}.$

228. $11x^2 = 200x + 171.$

229. Assuming that the first member of the following equation has three identically equal factors, find two values for a, and tell why you cannot find the other four:

$$a^6 - 18a^5 + 114a^4 - 288a^3 + 228a^2 - 72a + 8 = 185193.$$

230. Solve $\dfrac{1}{2x - 1} + \dfrac{3x + 4}{x + 2} = 2\frac{5}{6}.$

231. What value must be given to b in the following equation in order that when $a = 2$ the value of x may be 4?

$$\frac{2a^2 - b}{b - x} - \frac{a^2 - 2b}{x + b} + \frac{3a^2(b - x)}{25 - x^2} = 5.$$

232. $119 + \sqrt{4x^2 + 2x + 7} = 12x^2 + 6x.$

233. $9x - 7 + 2\sqrt{x^2 - 4x + 7} = x^2 + 5x.$

234. $12\left(\dfrac{x + 1}{x + 2} + \dfrac{x + 2}{x + 3}\right) = 17$

235. $x\left(\dfrac{x + 2}{x - 2} + \dfrac{x - 2}{x + 2}\right) = 2x + \frac{16}{3}.$

236. $3\left(1 + \dfrac{1}{7 - x}\right) = \dfrac{4}{x + 7}\left(1 + \dfrac{10}{x - 7}\right).$

237. $\dfrac{5}{4 - x} + \dfrac{2}{x - 2} + \dfrac{3}{x} = 0.$

238. $\frac{x^2}{2} - 3x = x + 3\sqrt{x^2 - 8x + 9}.$

239. $\frac{x}{x+1} + \frac{x+1}{x} = 2\frac{1}{56}.$

240. $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11.$

241. $(x^2 - 5x)^2 - 8(x^2 - 5x) = 84.$

242. $4x - \frac{14 - x}{x - 1} = 14.$ **243.** $\sqrt[3]{x^3 - 3x^2 + 1} = x.$

244. $3x^2y^2 - 7 = xy;\ x + 4xy = 9.$

245. $3x - \frac{2y - 5}{x - 2} = -\frac{4 - 9x}{3};\ 7x - 3y = 10.$

246. $x + y = 20;\ xy = 51.$

247. $x^2 + y^2 = 9;\ x + y = 3.$

248. $3(x^2 + xy) = 40y;\ x - y = 2.$

249. $x + y = x^2;\ 3y - x = y^2.$

250. $x - y = 5;\ \frac{1}{x} + \frac{1}{y} = \frac{1}{6}.$

251. $x^2 - 2xy + 3y^2 = 3;\ x^2 + 2xy + 3y^2 = 11.$

252. $29 = yz + \frac{y + z}{x};\ 14 = zx + \frac{z + x}{y};\ 2\frac{1}{3} = xy + \frac{x + y}{z}.$

253. $x^2 - 3y + \frac{1}{3y - x^2} = (2x + 5y - 33)(x^2 - 3y) = 0.$

254. $3x - 2y = 10;\ x(2y + 3) = 6$

255. $\frac{x - 3}{5} = \frac{y - 7}{2};\ 11x^2 = 2xy + 13y^2.$

256. $x^2 + 3xy = 12 - xy = 16y^2 - xy - x^2.$

257. $2x^2 + 3xy = 12 - xy + x^2 = 16y^2 - xy.$

258. $x^2 - 2xy + y^2 + 2(x + y) - 3$
$= x(x - y - 1) + y(x - y + 1) = 0.$

259. $xy + y^2 = y^2 - \frac{xy - x^2}{2} = 4.$

260. $\frac{1}{5}\left(xy + \frac{x}{y}\right) = \frac{xy^2 - x}{3y} = 2.$

261. $\frac{x}{y^2 - 3} = \frac{y}{3 - x^2} = \frac{7}{y^3 - x^3}.$

262. $\frac{x^2 + y^2}{17} = \frac{x^2 - y^2}{8}$; $25y^2 + 9x^2 = 450.$

263. $x^2 + 2xy - 4 = 7(2y^2 - 3xy) = 35.$

264. $\frac{3x^2}{4} - \frac{y^2}{3} = -4(2y - 3x) = 48.$

265. $3x^2 + 5xy - 2 = 11xy - 3y^2 + 1 = 20.$

266. $\frac{x}{y} + \frac{y}{x} = \frac{7}{2}$; $x^2 + 2y^2 = 54.$

267. $xy = 15$; $yz = 30$; $xz = 18.$

268. $y(y - 2) = x(x + 2)$; $y + 2 = x - 2.$

269. $x + 2y = 8$; $x^2 + 2y^2 = 22.$

270. $3\sqrt{(x + y)(x + 1)} = 3x + y + 2 = 4.$

271. $x^2 + y^2 - x - y = 78$; $xy + x + y = 39.$

272. $4x^2 + xy = 6$; $3xy + y^2 = 10.$

273. $x^2 + xy + y^2 = 52$; $xy - x^2 = 8.$

274. $x^2 + xy - 2y^2 = 7$; $x^2 - 9y^2 = 27.$

275. $(x - 2y)^2 + 3(x - 2y) + 2 = 0$; $x^2 - 2xy - 3x + 6y = 1.$

276. $2x^2 + y^2 = 24$; $xy = 8.$

277. $(x + 1)(y - 2) + (x + 1)^2 = 2$;
$(y - 2)^2 + 3(x + 1)(y - 2) = 4.$

278. $x + y = 12$; $x^2 + y^2 = 74.$

279. $+ y = a$; $x^2 - y^2 = b^2.$

280. $\frac{15}{x} : \frac{21}{y} = 3 : 7$; $x^2 - y^2 = 9.$

281. $2x^2 + 3xy - 3y^2 + 12 = 0$; $3x + 5y + 1 = 0.$

282. $\frac{3\sqrt{x} + 2\sqrt{y}}{4\sqrt{x} - 2\sqrt{y}} = 6$; $\frac{x^2 + 1}{16} = \frac{y^2 - 64}{x^2}.$

283. $91x^2 - 2x = 45.$ **284.** $x^4 - 21x^2 = 100.$

285. $x^{-3} - x^3 = 7(x^3 + 1).$

286. $x^4 - 2x^3 + x - 2 = 0.$

287. $(x + 1)(x - 2)(x^2 - 6x + 9) = 0.$

288. $\dfrac{2x - 5}{x + 2} = \dfrac{3}{5x}.$ **289.** $\dfrac{9x - 1}{x - \dfrac{1}{x}} = \dfrac{55}{6}.$

Form the quadratic equations whose roots are:

290. $(a - \frac{1}{2})$; $(b + \frac{2}{3})$.

291. $\dfrac{(a + b)^2}{a - b}$; $b - a.$

292. $-\frac{8}{3}$; $\frac{4}{7}$.

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293. One root of the equation $x^2 - 4x + c$ is $2 + \sqrt{3}$; what is the other root, and what is the value of c?

294. For what value of m will the equation

$$2x^2 + 3mx + 2 = 0$$

have equal roots?

295. Find the value of m which will make

$$x^2 - (3m - 1)x + 2m$$

exactly divisible by $x - 1$.

296. $\dfrac{\frac{1}{2}(x - a)}{b(x + a)} = \dfrac{1}{a} - 2\dfrac{b - \left(x - \dfrac{2b^2}{a}\right)}{(x + a)^2}.$

297. $\dfrac{a - c}{x - a} - \dfrac{x - a}{a - c} = \dfrac{3b(x - c)}{(a - c)(x - a)}.$

298. $x^4 + (2a^2 + 3ab - 2b^2)^2 = 5(a^2 + b^2)x^2.$

299. $x^2 - 5xy + y^2 = 15a^2;\ \dfrac{x}{2a} - \dfrac{y}{4a} = 1.$

300. $\dfrac{x}{a} + \dfrac{y}{b} = 1;\ \dfrac{a}{x} + \dfrac{b}{y} = -1.$

301. Solve the equations

$$\frac{2x}{y} = \frac{x+y}{x};\ x + y + a = 0;$$

separating the two sets of answers clearly from each other.

302. $\dfrac{1}{x} - \dfrac{1}{y} = 1;\ 2xy + 9 = 0.$

303. $\dfrac{ay}{x+a} - \dfrac{bx}{y-b} = \dfrac{a-b}{2};\ \dfrac{x}{a} - \dfrac{y}{b} = 2.$

304. $\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y} = \dfrac{a^2-b^2}{ab};\ x^2 + y^2 = 2(a^2+b^2).$

305. $\begin{cases} a(x-y) - b(x+y) = a^2 - b(2a+b); \\ (a+b)x + (a-b)y = a(a+2b) - b^2. \end{cases}$

306. One root of the equation $2x^3 + 3x^2 - 3x - 2 = 0$ is 1. Find the others.

307. Find all the roots of the equation $x^6 = (125)^2$.

308. What equation leads to the following answers: 1, 2, -3, 5?

309. $\dfrac{1}{y} - \dfrac{1}{x} = \dfrac{2}{3a};\ \dfrac{1}{a-x} - \dfrac{1}{a-y} = \dfrac{1}{4a}.$

310. $(a-b)x = (a+b)y;\ x + y = c.$

311. $\dfrac{x}{a} + \dfrac{1}{a}\cdot\dfrac{1+a}{x} = x + \dfrac{1}{a^2}.$

312. $abx^2 - (a^2+b^2)x + ab = 0.$

313. $\dfrac{x}{a} - \dfrac{y}{b} = 1;\ \dfrac{x}{b} + \dfrac{y}{a} = \dfrac{a}{b}.$

314. $(p+q)x-(p-q)y = 3pq;\ (p+q)x-(p-q)y = pq.$

315. $\dfrac{ax}{x-b} + \dfrac{bx}{x-a} = a + b.$

316. $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}.$

317. $\dfrac{ax^2-b}{ax+b}+\dfrac{a+bx^2}{a-bx}=\dfrac{2(a^2+b^2)}{a^2-b^2}.$

318. $(a-b)y^2-(a+b)y+2b=0.$

319. $\dfrac{1+p}{1-px}+\dfrac{1-p}{1+px}=1.$

320. $(ax-b)^2+4a(ax-b)=\frac{9}{4}a^2.$

321. $\dfrac{1}{b}+\dfrac{a}{x+ab}+\dfrac{a}{2x+ab}=0.$

322. $x^2-(a-b)x=(c-a)(c-b).$

323. $\dfrac{b}{x-a}+\dfrac{a}{x-b}-2=0.$

324. $\dfrac{bx+c}{ax+c}+\dfrac{cx+b}{ax+b}=\dfrac{(b+c)(x+2)}{ax+b+c}.$

325. $\dfrac{x}{a}+\dfrac{y}{b}=1;\ \dfrac{x}{3a}+\dfrac{y}{6b}=\dfrac{2}{3}.$

326. $\dfrac{bx}{c}-\dfrac{1}{c}\left(\dfrac{1}{a}+x\right)+d=\dfrac{d}{c}\left(cx-\dfrac{1}{ad}\right)-\dfrac{x}{c}+\dfrac{b}{c}.$

327. $\dfrac{a}{x-b}-\dfrac{b}{x+a}=\dfrac{a-b}{x}.$

328. $\begin{cases}(b+c)x+(b-c)y=2ab;\\(c+a)x+(c-a)y=2ac.\end{cases}$

329. $(b-c)x^2+(c-a)x+a-b=0.$

330. $bx+ay=a^2+b^2;\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{b^2}{a^2}+\dfrac{a^2}{b^2}.$

331. $\dfrac{a}{x}+\dfrac{b}{y}=2;\ by-ax=b^2-a^2.$

332. $a_1x-a_2y=2a_1a_2;\ 2a_2x+2a_1y=3a_2^{\,2}-a_1^{\,2}.$

333. $\dfrac{3\{pq-x(p+q)\}}{p+q}+\dfrac{(2p+q)q^2x}{p(p+q)^2}=\dfrac{qx}{p}-\dfrac{p^2q^2}{(p+q)^3}.$

334. $bx+y=x+ay=\frac{1}{2}(x+y)+1.$

335. $x + y = axy;\ y + z = byz;\ z + x = cxz.$

336. $a - yz = \dfrac{y+z}{x};\ y(b - zx) = z + x;\ z(c - xy) - y = x.$

337. $\dfrac{2a(a+b) - b^2x}{bx - 2a} = \dfrac{2}{\dfrac{b}{a^2}\left(\dfrac{1}{x} - \dfrac{b}{2a}\right)} - b\left[1 - \dfrac{b}{a - \dfrac{b}{2x}}\right].$

338. $\dfrac{2}{\dfrac{1}{b}\left(\dfrac{x}{2a} - 2\right)} - \dfrac{b}{a}\left[\dfrac{4a}{\dfrac{4b}{x} - \dfrac{b}{a}} - \dfrac{1}{\dfrac{1}{x} - \dfrac{1}{b}}\right] = 0.$

339. $\dfrac{1}{x+y} - \left(\dfrac{y}{a(x-y)} - \dfrac{x+6a}{x^2 - y^2}\right) = 0;$

$y : (7x - 2y) = (b - a) : (2a - 9b).$

[Substitute $a = 6$ and $b = -2$ in the answers.]

340. $\dfrac{\frac{1}{5}[2b(x+1)]^2}{4bx^3 + 5ax} - a\left(\dfrac{1}{x} - \dfrac{5ax - 4b}{4bx^2 + 5a}\right) = 0.$

341. $\dfrac{(a+2b)x}{a - 2b} = \dfrac{a^2}{a - 2b} - \dfrac{4b^2}{x}.$

342. $\dfrac{2}{1 + 3x} - \left[\dfrac{a(1+2x)}{b(1+3x)} - \dfrac{b(3x-1)}{a(2x+1)}\right] = 0.$

343. $\dfrac{x+1}{c} - \dfrac{2}{cx} = \dfrac{x+2}{ax - bx}.$ **344.** $\dfrac{abx}{a^2 + b^2} = 1 - \dfrac{a^2 - b^2}{(a^2+b^2)x}.$

345. $\dfrac{ax}{a^2x - 2} - \dfrac{1}{a}\left(\dfrac{x-3}{a^2x - 2} - \dfrac{1}{x}\right) = \dfrac{2}{2x - a^2x^2}.$

[Substitute $a = -1$ in the answers.]

346. $\dfrac{1}{x} = 2 - \dfrac{4ax^2 - 3b(x-2)}{2a(x^2+1) + 3b}.$

347. $a^2\dfrac{2x-1}{x+2} = b^2\dfrac{x+2}{2x-1}.$

348. $ax - \dfrac{4}{ax + b}[b^2(1+x)x - a^2(1-x)] = b.$

349. $a\left(\dfrac{x}{x+3a}-\dfrac{2b}{x-a}\right)=\dfrac{b}{2}-\left(\dfrac{a}{2}-\dfrac{bx}{x-a}\right).$

350. $\dfrac{2ax-4b}{bx-a}-\dfrac{bx-a}{2ax-b}=\dfrac{2abx}{2abx^2-(2a^2+b^2)x+ab}.$

351. $\dfrac{1}{1+a+x}=1+\dfrac{1}{a}+\dfrac{1}{x}.$

352. $ax-\dfrac{(2b+a)x+a}{x+1}=\dfrac{b^2(x-1)}{4a}.$

353. $\dfrac{x}{m^2p(x+a)}=\dfrac{x+a}{n^2px}.$

354. $\dfrac{2b-x-2a}{bx}=\dfrac{x-4a}{ab-b^2}-\dfrac{4b-7a}{ax-bx}.$

355. $\dfrac{2a-x-19b}{ax-2bx}=\dfrac{a-2b-x}{a^2-4b^2}-\dfrac{5b-x}{ax+2bx}.$

356. $\left(1-\dfrac{1}{x}\right)\left(1-\dfrac{x}{a}\right)-\dfrac{1}{a}\left(\dfrac{x}{a}-\dfrac{2a^2-3x}{x}\right)=0.$

357. $(a+1)\dfrac{x-2}{x-1}-\dfrac{a}{a+1}\left[2+\dfrac{1}{x}-\dfrac{(a+1)^2}{a(x-1)}\right]=0.$

358. $\dfrac{x-3b}{a-3b}-\left[\dfrac{2x}{x-3b}-\dfrac{3b(x+3b)}{2a(a-3b)}\right]=0.$

359. $\dfrac{x+b}{2a}+\dfrac{2a}{x-b}=1-\dfrac{2a}{b}\left(1-\dfrac{2a-b}{x-b}\right).$

360. $(x+a)(x-b)-\dfrac{a^2(x+a)}{x+b}-\dfrac{b^2(x-b)}{x-a}=\dfrac{3a^2b^2}{(x-a)(x+b)}$

361. $(x+3y):(2x-y)=\left(\dfrac{1}{b}-\dfrac{7}{3a}\right):\dfrac{2}{b};$

$$x^2=\tfrac{1}{2}(xy+3ay+18a^2).$$

[Substitute $a=2$ and $b=-3$ in the answers.]

362. $\dfrac{a}{y+4b}=\dfrac{2b}{x-y};\ \dfrac{1}{(b-a)x}-\left[\dfrac{3}{(a+b)y}-\dfrac{1}{a^2-b^2}\right]=0.$

[Substitute $a=3$ and $b=-1$ in the answers.]

363. $\frac{x}{a}+\frac{y}{b}=1;\ \frac{a}{x}+\frac{b}{y}=4.$

364. $(3+b^2)(x^2-x+1)=(3b^2+1)(x^2+x+1).$

365. $\frac{(4a^2-b^2)(x^2+1)}{4a^2+b^2}=2x.$

366. $\frac{4a^2}{x+2}+\frac{4a^2-b^2}{x(x^2-4)}=\frac{b^2}{x-2}.$

367. $\frac{a}{x}+\frac{b}{y}=p;\ \frac{b}{x}+\frac{a}{y}=q.$

368. $\frac{2x+1}{b}-\frac{3x+1}{a}=\frac{1}{x}\left(\frac{1}{b}-\frac{2}{a}\right).$

369. $$\frac{a}{b(2x-1)}-\frac{b(2x+1)}{a(x^2-1)}$$
$$=\frac{1}{(2x-1)(x+1)}+\frac{1}{(2x-1)(x-1)}.$$

370. $\frac{x+3b}{8a^2-12ab}+\frac{3b}{4a^2-9b^2}=\frac{a+3b}{(2a+3b)(x-3b)}.$

371. $ax+by=p;\ cy+dz=q;\ ex+fz=r.$

372. $\frac{x+13a+3b}{5a-3b-x}-\frac{a-2b}{x+2b}=1.$

373. $$\frac{a+1}{(a+2)x-(a+3)}-\frac{a+4}{(a+5)x-(a+6)}$$
$$=\frac{b+1}{(b+2)x-(b+3)}-\frac{b+4}{(b+5)x-(b+6)}.$$

374. $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}.$

375. $\frac{a}{x}+\frac{y}{b}=2;\ xy+ay=bx+ab.$

376. $ax-by=0;\ px-qy=m.$

377. If a, b, c, and d are in G. P., prove that

$$\frac{(ac-bd)(ab-cd)}{(ac+bd)(ab+cd)}+\frac{2ac}{a^2+bd}=1.$$

378. If a, b, c, d are in proportion, prove that

$$a^2 + b^2 - \frac{b^2c}{d} = \frac{a^2c + b^2d}{c + d}$$

$$a^2 : c^2 = \frac{a^2x + aby + b^2z}{c^2x + cdy + d^2z}.$$

379. If $a : b = x : y$, prove that $a^2 + b^2 : a(a - b) = \dfrac{x^2 + y^2}{x(x - y)}$.

380. If a, b, p, q are in proportion, prove that

$$ma + nb : ax - by = pm + qn : px - qy;$$

if they are in CONTINUED proportion, prove that

$$\left(\frac{a^3 + b^3}{b^3 + p^3}\right)^2 = \frac{p^6}{q^6}.$$

1. Show that if $\dfrac{x}{a + 2b + c} = \dfrac{y}{2a + b - c} = \dfrac{z}{4a - 4b + c}$,
then $\dfrac{a}{x + 2y + z} = \dfrac{b}{2x + y - z} = \dfrac{c}{4x - 4y + z}$.

382. If $c : d = x : y$, show that $\dfrac{cd}{xy} = \dfrac{c^2 + d^2}{x^2 + y^2}$.

383. Insert two geometric means between a^{6x} and b^{6x}.

384. If b, c, and $2b - a$ are in G. P., show that ab, b^2, and c^2 are in A. P.

385. Prove that if $\dfrac{2x + 3y}{2x - 3y} = \dfrac{2a + 3b}{2a - 3b}$, then a, b, c, and d are in proportion.

386. What relation exists between the values of x and y in the equation $(12x + y) : (11x + y) = 9 : 7$?

387. The first and second terms of an H. P. are respectively 70 and 60; find the fifth and seventh terms.

388. An H. P. has the same first two terms as a G. P. whose common ratio is $\frac{1}{6}$. Find the ratio of their third terms.

389. Insert c arithmetical means between a and b.

390. Find the seventh term of $2;\ \frac{3}{2};\ 1;\ \ldots$

391. Find the sixth term of $2;\ \frac{3}{2};\ 1\frac{1}{8};\ \ldots$; also the sum of the first six terms.

392. Given the series $3\frac{3}{4} + 1\frac{1}{4} + \frac{5}{12} + \ldots$, find the sum of the first five terms and of the whole series.

393. Find the sum of nine terms of the series $2\frac{1}{2},\ 3\frac{3}{4},\ 5 \ldots$

394. The first and third terms of a G. P. are 4 and $2\frac{1}{4}$ respectively; find the sum of the whole series.

395. Find the sum of n terms of the series

$$(1 - \sqrt{2}) + (2 + \sqrt{2}) + (3 + 3\sqrt{2}) + \cdots$$

396. Find the sum of n terms: $7 + 1 + \frac{1}{7} + \ldots$

397. Find the tenth term, the sum of ten terms, and the entire sum of the series whose second term is 5 and whose sixth term is $\frac{1}{125}$.

398. Find the sum of n terms of an A. P. of which the first two terms are a and b; also of the G. P. which begins with the same two terms.

399. The sum of five numbers in continued proportion is 242, and the ratio of the third term to the fourth is 3; find the first.

400. If a, x, and y are in continued proportion, and if a, $x + a$, $y + a$ are in A. P., prove that $a = \dfrac{x + a}{3} = \dfrac{y + a}{5}$.

401. How many terms of the series 18, 15, 12, . . . amount to 60?

402. If there are five numbers such that the first three and the last three are in A. P., while the first, third, and fifth are in G. P., prove that the product of the first and fourth, added to the product of the second and fifth, will be double the product of the second and fourth.

403. The sum of the squares of the successive whole numbers beginning with 1 and ending with a certain number is 20 times that number; find the number.

404. Find the sum of all multiples of 7 between 50 and 450.

405. Find the sum of all multiples of 3 between 1 and $3a + 1$.

406. Sum the G. P. $27; -9; \ldots$ to infinity.

407. Sum the series:

I.	$30 + 15 + 7\frac{1}{2} + 3\frac{3}{4} \ldots$	to 10 terms.
II.	$30 + 15 + 0 - 15 \ldots$	to 10 terms.
III.	$2.7 + .09 + .003 + .0001 + \ldots$	to 10 terms.

If each of these series be continued to infinity, which will have a finite sum? If there is any such, find the sum.

408. Insert $2m - 1$ arithmetic means between a and b.

409. Identify the series $\frac{1}{\sqrt{2}}$; $\frac{1}{1 + \sqrt{2}}$; $\frac{1}{4 + 3\sqrt{2}}$; and write its fourth term.

410. The first 121 integers can be arranged in five groups, each of which is a perfect square, thus: (1); (2, 3, 4); (5, . . . 13); (14, . . . 40); (41, . . . 121). Prove this, and extend the property to a sixth group.

411. The second term of a G. P. is 54, and the fifth term 16; find the series, and its sum.

412. Find the sum of the first n natural numbers.

413. Find three geometrical means for 2 and 162.

414. Find the sum to infinity of the G. P. $\frac{3}{4}; \frac{1}{2}; \frac{1}{3}; \ldots$

415. Find the sum of 18 terms of the series $\frac{2}{3}; -1; -2\frac{2}{3}; \ldots$

416. A person saves $270 the first year, $210 the second, $150 the third, and so on; in how many years will a person who saves every year $180 have saved as much as he?

417. Insert three geometrical means between $3\frac{5}{9}$ and 18.

418. The first term of an A. P. is 2, and the difference between the third and seventh terms is 6. Find the sum of the first 12 terms.

419. There are two numbers whose geometric mean is $\frac{4}{5}$ of their arithmetic mean; and if the two numbers be taken for the first two terms of an arithmetic progression, the sum of its first three terms is 36. Find the numbers.

420. There are two numbers whose arithmetic mean is $33\frac{1}{3}\%$ greater than their harmonic mean. Find their ratio.

421. The sum of three terms in A. P. beginning with $\frac{3}{2}$ is equal to the sum of three terms in G. P. beginning with $\frac{3}{2}$, and the common difference in the first case is equal to the common ratio in the second. What are the two series?

422. Find the sum of ten terms of the G. P. in which the fourth term is 1 and the ninth term is $\frac{1}{243}$.

423. Find the ratio of an infinite geometrical series of which the first term is 1 and the sum of the terms $\frac{5}{4}$.

424. Find the sum of the arithmetical series formed by inserting nine means between 9 and 109.

425. The first and ninth terms of an A. P. are 5 and 22; find the sum of 21 terms.

426. What is the sum of the first 200 odd numbers?

427. Find the sum of

$$1 + (1 + b) + (1 + 2b) + (1 + 3b) + \dots + (1 + nb)$$

when $b = 2$, $n = 11$.

428. A and B start at the same time from the same point in the same direction. A goes at the uniform rate of 60 miles per day; B goes 14 miles the first day, 16 miles the second day, 18 miles the third day, and so on. At the end of 50 days who will be ahead, and how much?

429. A traveller has a journey of 132 miles to perform. He goes 27 miles the first day, 24 the second, and so on, travelling 3 miles less each day than the day before. In how many days will he complete the journey?

430. If $x - y$ is a mean proportional between y and $y + z - 2x$, show that x is a mean proportional between y and z.

431. Find three numbers in geometrical progression such that their sum shall be 14 and the sum of their squares 84.

432. What is the geometrical mean between $2x - 3$ and $2x^3 + x^2 - 4x - 3$?

433. If the arithmetic mean of two numbers is $\frac{15}{2}$ and their geometric mean is $\frac{9}{2}$, find the numbers.

434. Reduce $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$ to equiradical surds, and find their sum, difference, product, and quotient.

435. Square root of $44 - 16\sqrt{7}$.

436. Square root of $3(a - 1) + 2\sqrt{2a^2 - 7a - 4}$.

437. Find the value of

$$(35\sqrt{ac} + 77\sqrt{a} + 63\sqrt{b} + 28\sqrt{ab})(\sqrt{ac} - \sqrt{a} - \sqrt{b})$$

when $a = 2$, $b = 3$, $c = 5$; and simplify the result.

438. Square root of

$$\frac{9a^{2a}c^2}{4b^{12}} - \frac{3a^{a+b}c}{b^3} + a^{2a}b^6 - \frac{2^8a^ac}{b^6} + \frac{2^9a^bb^3}{3} + \frac{2^{16}}{9}.$$

439. Simplify $\sqrt{-a}$ when $\dfrac{47 + a}{12} = \sqrt{15}$.

440. When $x = \sqrt{2}$ find the value of $\dfrac{2x - 1}{(x - 1)^2} - \dfrac{2x + 1}{(x + 1)^2}$.

441. $\dfrac{\sqrt{12}}{(1 + \sqrt{2})(\sqrt{6} - \sqrt{3})}$.

442. $\sqrt{4 + x} + \sqrt{6 + 2x} - \sqrt{6 - x} = 0$.

443. $\frac{1}{2}\sqrt{2x + 8} + \sqrt{x + 5} = 1$; verify your result.

444. $(\sqrt[3]{x^7}) \times (\sqrt[5]{x^9}) \times x^{-\frac{1}{3}} \div x^{\frac{4}{5}}$.

445. $x\sqrt{x^2 + 12} + x\sqrt{x^2 + 6} = 3$.

446. $\sqrt{3x+1} = \sqrt{4x+5} - \sqrt{x-4}$.

447. $2\sqrt[3]{40} + 3\sqrt[3]{108} + \sqrt[3]{500} - \sqrt[3]{320} - 2\sqrt{1372}$.

448. Square root of

$$1 + 4y^{-\frac{1}{3}} - 2y^{-\frac{2}{3}} - 4y^{-1} + 25y^{-\frac{4}{3}} - 24y^{-\frac{5}{3}} + 16y^{-2}.$$

449. Simplify $\sqrt{\frac{1}{2}} : \sqrt{\frac{1}{3}}$; also $(a^{x-3})^x : \left(\frac{\sqrt[8]{b}}{a}\right)^{x+1}$.

450. $\left(\frac{a^{\frac{1}{2}}b}{c^{-\frac{3}{4}}}\right)^{\frac{1}{3}} \times \left(\frac{\sqrt{c}}{e^{\frac{4}{3}}}\right)^{-\frac{1}{2}} \div \left(\frac{e}{b}\right)^{\frac{3}{3}}$.

451. Simplify $(a^m)^n$; $\frac{a^0}{b^{-m}}$; $(-a)^{2n}(-a)^{2n+1}$; $\sqrt{a^{2m}}$;

$\sqrt{108} + \sqrt{75} - \sqrt{27}$; $5^{\frac{3}{2}} + 3 \,.\, 5^{\frac{1}{2}}$; $(2^{\frac{2}{3}} \,.\, 2^{\frac{1}{2}})^{\frac{6}{7}}$.
[Suppose n to be an integer.]

452. Find the value of $\sqrt{\frac{5}{x}} - \sqrt[3]{-x}$ when $x = .008$.

453. $\sqrt{x+a} + \sqrt{x} + \sqrt{x-a} = 0$; explain the possibility of satisfying this equation, the connecting signs both being plus.

454. $\frac{a^m \times a^n}{a^{2m} \times a^{-m}} \div \frac{a^m \div a^n}{a^{2m} \times a^{-m}}$.

455. $(2-\sqrt{6})\left(\frac{1}{\sqrt{5}-\sqrt{2}-\sqrt{3}} - \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}\right)$.

456. $11 + 3\sqrt{\frac{x+2}{x}} = 4\sqrt{\frac{x}{x+2}}$; verify your results.

457. Square root of $a^{-2} + 2a^{-1}(2-b^{-2}) + b^{-4} + 4(1-b^{-2})$.

458. Square root of $49a^4 - 28a^3 - 17a^2 + 6a + \frac{9}{4}$.

459. $7\sqrt{x-8} - 2\sqrt{3} = \sqrt{3(7x+4)}$. **460.** $\sqrt{87 - 12\sqrt{42}}$.

461. Find the value of

$$(x^{\frac{1}{4}}a + y^{\frac{1}{4}}b)(y^{\frac{1}{4}}a - x^{\frac{1}{4}}b) - (xy)^{\frac{1}{4}}(a^2 - b^2) + x^{\frac{1}{2}}ab$$

when $x = 2$ and $y = 3$.

462. Given $(\sqrt{8} - 1) : 4 - a = 4 + a : 3$, find a.

463. Find the square root of

$$\frac{4a^4}{b^2} - 12a^3 + 9a^2b^2 - 8ab + 12b^3 + \frac{4b^4}{a^2}.$$

464. $\left(\dfrac{x^{-\frac{1}{2}}y^{-\frac{1}{3}}}{x^{-\frac{1}{4}}y^{-\frac{5}{6}}} \div \sqrt[4]{x^{-3}y^{-5}}\right)^{\frac{3}{7}}$. **465.** $\dfrac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{x+3} - \sqrt{x-3}}$.

466. Square root of $x^{\frac{2}{3}} + 9x^2 - 4x + 10x^{\frac{4}{3}} - 12x^{\frac{5}{3}}$.

467. $(2 - \sqrt{2} + \sqrt{3})(1 + \sqrt{2} + \sqrt{6})$.

468. Given $x - \sqrt{9 + x\sqrt{x^2 - 3}} = 3$; find x.

469. Square root of $a^2b^{-2} - 10ab^{-1} - 10a^{-1}b + a^{-2}b^2 + 27$.

470. $\dfrac{a^3 - b^4}{a^{\frac{1}{2}} - b^{\frac{2}{3}}}$. **471.** Value of $\left(\sqrt[12]{32}\right)^{-3}$.

472. Solve $\dfrac{1}{1 + \sqrt{1 - x}} + \dfrac{1}{1 - \sqrt{1 - x}} = \dfrac{3}{7}x + 5$.

473. Reduce

$$\sqrt{2a^2 + ab - 6b^2}\,\sqrt{3a^2 + 5ab - 2b^2}\,\sqrt{6a^2 - 11ab + 3b^2},$$

and find its value when $a = 3$ and $b = -1$.

474. Simplify $x^{-\frac{1}{2}}y + x^{\frac{3}{2}}y^{-1} - x^{\frac{1}{2}}y^{-\frac{2}{3}}$, expressing it as a single fraction.

475. $\dfrac{a^{-\frac{m}{n}}b^{\frac{1}{m}} + a^{\frac{1}{n}}b^{-\frac{n}{m}}}{ab}$

476. $\sqrt{x - 4} + \sqrt{x - 11} - \sqrt{2x + 9} = 0$.

477. $$\frac{\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} + \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}}{\dfrac{x^2 + y^2}{x^2 - y^2} + \dfrac{\frac{2}{x - y}}{\frac{1}{x} + \frac{1}{y}}}.$$

478. Find the value of $4y - x$ when

$$x = 1 + \frac{2\sqrt{3}}{2 - \sqrt{3}}; \qquad y = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}.$$

479. $\sqrt{5 - 2x} + \sqrt{15 + 3x} = \sqrt{26 - 5x}.$

480. $a^m b^n \times a^{2m} b^{-2n} \div (a^{3m} b^{-2n} \times \sqrt{b^{2n}}).$

481. Rationalize the denominator of $\dfrac{\sqrt{-3} + 3}{\sqrt{-4 - 2\sqrt{3}}}.$

482. Find the value of $\dfrac{2\sqrt{7} - 2\sqrt{343} + 7\sqrt{28}}{6\sqrt{63}}$ to two decimal places.

483. $\left[\dfrac{\sqrt[3]{a}}{\sqrt[4]{b}}\left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right]^4.$

484. $\sqrt{3 + x} + \sqrt{x} = \dfrac{5}{\sqrt{x}}.$

485. Express with a rational denominator $\dfrac{x^2}{y + \sqrt{y^2 - x^2}}.$

486. State the value of

$$(x^{\frac{2}{3}})^{\frac{1}{2}}; \quad (y^{-\frac{3}{4}})^0; \quad \frac{a}{a^{-\frac{2}{5}}}; \quad (-27)^{\frac{4}{3}}; \quad (\tfrac{1}{9})^{-\frac{1}{2}}.$$

487. Find the square root of

$$x - 2x^{\frac{1}{2}}y^{-\frac{1}{4}} + 8ax^{\frac{3}{2}} + y^{-\frac{1}{2}} - 8axy^{-\frac{1}{4}} + 16a^2x^2.$$

488. $(7 - 4\sqrt{3})x^2 + (2 - \sqrt{3})x = 2.$

489. Find the value of $2a\sqrt{1 + x^2}$ when

$$x = \frac{1}{2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right).$$

490. $xy^{-2} - 2x^{\frac{1}{2}}y^{-1}z^{-\frac{1}{2}} + z^{-1}.$

491. $4\sqrt[4]{x} + \sqrt{x} = 21$; verify your answers.

492. $\dfrac{a^{m-\frac{1}{2}}}{b^{m+\frac{1}{2}}} \div \dfrac{a^{m+\frac{1}{2}}}{b^{m-\frac{1}{2}}}$.

493. $(256x^{\frac{2}{3}}y^{-\frac{4}{3}})^{-\frac{5}{4}}$.

494. $\sqrt{x^{\frac{3}{5}}} = 2\sqrt{2}$.

495. Value of $\dfrac{1}{\sqrt{3}+1}$ to three decimal places.

496. $\sqrt{1 + x - x^2} - 2(1 + x - x^2) = \frac{1}{9}$.

497. Multiply $x - \sqrt{5} + 1 - \sqrt{-10 - 2\sqrt{5}}$

by $x - \sqrt{5} + 1 + \sqrt{-10 - 2\sqrt{5}}$.

498. $\sqrt{13 + x} + \sqrt{13 - x} = 6$.

499. $x^2 - ax : \sqrt{x} = \sqrt{x} : x$.

500. $[x - \frac{1}{2}(1 - \sqrt{-3})][x - \frac{1}{2}(1 + \sqrt{-3})]$.

501. $a + x = \sqrt{a^2 + x(b^2 + x^2)^{\frac{1}{2}}}$.

502. $\dfrac{x^2y^{-\frac{2}{3}} - 2 + x^{-2}y^{\frac{4}{3}}}{x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}}}$.

503. $(\sqrt{-a} + c\sqrt[3]{b})(\sqrt{-a} - c\sqrt[3]{b})$.

504. Rationalize the denominator of $\dfrac{\sqrt{x} - \sqrt{x+y}}{\sqrt{x} + \sqrt{x+y}}$.

505. $(x - 5 + 2\sqrt{-1})(x - 5 - 2\sqrt{-1})$.

506. $\dfrac{1 - a^{-2} - y^2}{1 - x^{-3}y^{-2} + x^{-2}}$.

507. $(a^{\frac{5}{2}} - a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}})(a^{\frac{1}{2}} + b^{\frac{1}{3}})$.

508. $\dfrac{x^{-\frac{1}{2}} + a(x^2 + a^2)^{-\frac{1}{2}}}{\sqrt{a^2 + x^2} + a\sqrt{x}}$.

509. $\left(x - \dfrac{1 - \sqrt{3}}{2\sqrt{2}}\right)\left(x - \dfrac{1 + \sqrt{3}}{2\sqrt{2}}\right)\left(x + \dfrac{1}{\sqrt{2}}\right)$.

510. $\left(\dfrac{10\sqrt[3]{a^2}}{3\sqrt[4]{b^5}}\right)^2 : x = \sqrt{\dfrac{5a\sqrt[3]{a^2}}{4\sqrt[5]{a^2b^9}} : \dfrac{9b^{-3}}{\sqrt{5}}}$. Find x and simplify the answer.

511. Divide $6x^{m+3n} - 19x^{m+2n} + 20x^{m+n} - 7x^m - 4x^{m-n}$ by $3x^{2n} - 5x^n + 4$.

512. $(2x-1)^{\frac{1}{2}} - (3x+1)^{\frac{1}{2}} = (x-4)^{\frac{1}{2}}$.

513. $\sqrt[26]{\left[\sqrt[3]{x^2} \cdot \sqrt{\left(\dfrac{\sqrt{x}}{\sqrt[3]{x}}\right)^5}\,\right]^3}$.

514. $\sqrt{134 + 84\sqrt{2}}$.

515. $\dfrac{5ac}{b^2}\sqrt[3]{ab^2} : \sqrt[4]{\dfrac{9c^3}{a^2}} = x : \dfrac{3a^2}{2}\sqrt{\dfrac{3c}{ab}}$.

516. $\sqrt{7-x} + \sqrt{3x+10} + \sqrt{x+3} = 0$.

517. $x^{\frac{1}{2}} - a^{\frac{1}{2}} = (x-b)^{\frac{1}{2}}$.

518. Which is the greater, $\sqrt{10}$ or $\sqrt[5]{46}$, and why?

519. Square root of $75 - 12\sqrt{21}$.

520. Solve $(x-a)^2 = (x-2a)(x^2+4a^2)^{\frac{1}{2}}$.

521. Square root of $41 + 12\sqrt{5}$.

522. $\sqrt{x + \frac{1}{3}} = \sqrt[3]{x + \frac{1}{6}}$.

523. Reduce $\dfrac{5 + 3\sqrt{-1}}{1 + \sqrt{-1}}$ to the form $A + B\sqrt{-1}$.

524. $\sqrt{2x-y} = \sqrt{x-y} + 1$; $x^2 + 4y^2 = 17$.

525. $(-1 + \sqrt{-3})^3 + (-1 - \sqrt{-3})^3$.

526. $(\sqrt{3} + \sqrt{-2})(\sqrt{3} - \sqrt{-2})$.

527. Find the reciprocal of $\left(\dfrac{\sqrt{x^n+2} + \sqrt{x^n-2}}{2}\right)^2$.

528. Multiply $2\sqrt[3]{a} - \sqrt{-x}$ by $3\sqrt{-a} + 2\sqrt[3]{x}$.

529. Square root of $35 - 12\sqrt{6}$.

530. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}=12.$

531. Rationalize the denominators of

$$\frac{ac}{3a^{\frac{1}{2}}b^{\frac{3}{4}}c^{-\frac{1}{3}}};\quad \frac{1}{a^{\frac{2}{3}}+b^{\frac{1}{2}}};\quad \frac{7+2\sqrt{6}}{9-3\sqrt{6}}.$$

532. Simplify

$$\left(\frac{a^{-5}b^{-2}c^2}{a^{-3}b^3c^4}\right)^{-2};\quad \frac{x^{\frac{1}{2}}y^{-\frac{3}{2}}}{x^{\frac{2}{3}}y^{\frac{1}{4}}};\quad \frac{a^{2n+1}b^{n-1}}{a^{n+2}b^{1-n}};\quad \sqrt{\frac{2}{\sqrt[3]{2}}}.$$

533. $\sqrt{11+4\sqrt{6}}.$ **534.** $\sqrt{5-\sqrt{24}}.$

535. $\dfrac{\sqrt{a+x}}{\sqrt{a}+\sqrt{a+x}}=\dfrac{\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}.$

536. $\sqrt{\dfrac{x}{a^3}}\times\sqrt[3]{\dfrac{a}{y}}\times\sqrt[6]{\dfrac{y^4}{a^{-1}x}}.$

537. $\sqrt{\dfrac{ay}{x}}\times\sqrt[3]{\dfrac{bx}{y^2}}\times\sqrt[4]{\dfrac{x^3y^5}{{}^{-\frac{1}{3}}b^2}}.$ **538.** $\dfrac{1}{a}\div\sqrt{\dfrac{\sqrt[10]{a^6}}{\sqrt{a}(\sqrt[3]{a}\sqrt[5]{a})^2}}.$

539. Find the value of $\dfrac{a^{\frac{3}{2}}+ab}{ab-b^3}-\dfrac{\sqrt{a}}{\sqrt{a}-b}.$

540. $\sqrt{2+\sqrt{3}}.$

541. Given $x=3-\sqrt{11}$, $y=3+\sqrt{11}$; find the values of the expressions xy; $\dfrac{1}{x}-\dfrac{1}{y}.$

542. $\sqrt{x}+\sqrt{4a+x}=2\sqrt{b+x}.$

543. $(2\sqrt{-3}+3\sqrt{-2})(4\sqrt{-3}-5\sqrt{-2}).$

544. Find one value of $\sqrt[4]{17\quad 12\sqrt{2}}.$

545. Develop by binomial formula

$$(a^2+1+a^{-2})^3.$$

546. Expand $(a^2 - 2b)^3$.

547. Convert $\dfrac{1}{\sqrt{1+x^2}}$ into an infinite series by the binomial theorem.

548. Write down the eighth term of $(a - b)^{12}$.

549. Expand to four terms $\sqrt{a - x^2}$.

550. Expand to five terms $(1 + a)^{xy}$.

551. Expand $\dfrac{1}{(2a - 3)^{\frac{1}{3}}}$ into a series.

552. Expand to four terms $(a + x)^{-\frac{1}{4}}$.

553. Expand by the binomial theorem $3b(2x - y)^{\frac{1}{2}}$.

554. Write out the first five terms and the last five terms of $(x - y)^{13}$.

555. Calculate the sixth term of

$$\left(\frac{\sqrt[3]{a}}{\sqrt[4]{2}\,\sqrt[11]{b^8}} - \frac{\sqrt{2}}{3a^4\,\sqrt[10]{b}}\right)^{27}.$$

556. Find the fourth term of $\left(\dfrac{2\sqrt{a}}{3} - \dfrac{6\sqrt[3]{b^2}}{a}\right)^{21}$.

557. Find the sixth term of $\left(\dfrac{a\sqrt{a}}{\sqrt[9]{b^2}} - 6\sqrt{b^3}\right)^{17}$.

558. Find the last four terms of $(a^{\frac{1}{2}} - 2b^{\frac{1}{3}})^{20}$.

559. Find the terms which do not contain radicals in the development of $\left(\sqrt{2a} - \sqrt{\dfrac{b}{a}}\right)^4$.

560. Find the prime factors of the coefficient of the sixth term of the nineteenth power of $(a - b)$. What are the exponents in the same term, and what is the sign?

561. Find the sixth term of $(x - y)^7$; also of

$$\left(\frac{6a^2}{7b\sqrt{b}} - \frac{b}{\sqrt{3a}}\right)^7$$

562. Find the sixth term of the nineteenth power of

$$\left(\sqrt[3]{x^2} - \frac{y^3}{2x}\right).$$

563. Find the tenth term of $(x - y)^{27}$; also of

$$\left(\frac{9a}{\sqrt{b}} - \frac{2b}{\sqrt{a}}\right)^{27}.$$

564. Write out $(x - y)^{11}$.

565. Write out the first five terms and the last five terms of $(x - y)^{31}$; then find and simplify the fifth term of

$$\left(a^3b - \frac{3b^{-2}}{\sqrt{a^5}}\right)^{31}.$$

566. Find the sixth and twenty-fifth terms of the twenty-ninth power of $(x - y)$; also the sixth term of the twenty-ninth power of $\left(\frac{\sqrt{a}}{b} - \frac{b^2}{2a}\right)$.

567. Find the term which contains x^4 in $(1 - \frac{3}{2}x)^{\frac{2}{3}}$.

568. Find the middle term of $(a - 2x)^{12}$.

569. Find the middle term of $(1 + x)^{2n}$.

570. Write down the coefficient of x^p in the expansion of $\dfrac{1}{\sqrt[n]{(1 + x)}}$.

571. Expand $(1 - x^2)^{-5}$ to six terms by the binomial theorem.

572. Find the sixth term and the pth term in the expansion of $(a - x)^{2n}$.

573. Find the ratio of the nth term of $\dfrac{1}{(1 - x)^n}$ to the

nth term of $\left[\frac{(1+x)^n}{1+x}\right]^2$, as expanded by the binomial theorem.

574. Find the eleventh term in the expansion of

$$(3^9 + 3^6x)^{\frac{7}{3}}.$$

575. Find the fifth term of $(a^{-\frac{1}{3}} + 2x^{-\frac{1}{3}})^{-\frac{1}{5}}$.

576. Expand $(1-y)^{-5}$ to five terms, and write down the $(r+5)$th term in its simplest form.

577. If A is the sum of the odd terms, and B of the even terms, in the expansion of $(x+a)^4$, show that

$$A^2 - B^2 = (x^2 - a^2)^4.$$

578. Write down the first four terms, the last four terms, and the middle term of $(x-2y)^{14}$.

579. Expand by the binomial theorem $\left(x + \frac{1}{x}\right)^7$

580. Expand $(m^{-\frac{3}{5}} + 2n^3)^7$.

581. Find the fifth term of $(x^{-1} - 2y^{\frac{1}{2}})^{11}$.

582. Give the first, third, and fifth terms in the expansion of $\left(x\sqrt{y} + \frac{y^2}{2\sqrt{x}}\right)^{10}$.

583. Find the first term with a negative coefficient in the expansion of $(1+x)^{\frac{11}{3}}$.

584. Fifth term of $\left(\frac{3a^2}{\sqrt[3]{b}} - \frac{\sqrt[5]{a}}{3^3b}\right)^7$.

585. Fourth term of $\left(\frac{\sqrt{a}}{\sqrt[7]{b^2}} - \frac{b^3}{5a^5}\right)^{31}$.

586. Fourth term of $\left(\frac{x}{2\sqrt{y^2}} - \frac{3\sqrt[4]{y^3}}{x^2}\right)^9$.

587. Find the fourth term of $(x^2 - 2)^{13}$.

588. Find the fourth term of the eleventh power of

$$\left(\frac{1}{3}\sqrt[3]{b}-\frac{3a^2}{5\sqrt{b}}\right).$$

589. Find the eighth term of $\left(\frac{4\sqrt[4]{b^3}}{3a^5}-\frac{1}{2}a\sqrt{a}\,.\,b^{-\frac{1}{3}}\right)^9.$

590. Write out $(x-y)^{11}$; then find, in its reduced form, the *fourth term* of this expansion, when

$$x=\frac{\sqrt[4]{a^3}}{\sqrt{b}} \quad \text{and} \quad y=\frac{b\sqrt[3]{b^2}}{3a}.$$

591. Find the eleventh term in the expansion of $(2^8+2^6x)^{\frac{11}{2}}$.

592. Given log 1.3287 = .1234269 and log 1.3288 = .1234596; find log .00132874.

593. Given log 3.8795 = .5887758 and log 38796 = 4.5887870; find log $(\sqrt{.0387957})(\sqrt{387.954})$.

594. Find the value of $7\log_2\frac{16}{15}+5\log_2\frac{25}{24}+3\log_2\frac{81}{80}$.

595. Find the number whose logarithm is − 2.4211522, having given log 379.18 = 2.5788454 and log 379.19 = 2.5788569.

596. Find the logarithm of 6561 to the base $3\sqrt{3}$.

597. Given log 3000 = 3.4771213, solve the equation $(.01)^x = .0009$.

598. Given log 2 = .30103 and log 3 = .47712; find the values of $\log\sqrt[5]{3.6}$ and $\log_6\sqrt[5]{3.6}$.

599. Given log 2.5 = .39794 and log 2.25 = .35218; find log 2.7 to four places.

600. Given $\log a = 2$, $\log b = 3$; find $\log_a b^2$.

Practice in Forming Equations.

408. In forming the equation for any problem where the selection of a meaning for x seems difficult, it is a good plan to write down a list of the different numbers which would be mentioned in an explanation of the problem, but whose numerical value is not given in the statement of the problem. Then let x stand for one of these, and see if abbreviations for the others can be readily derived; perhaps it will be necessary to have more than one letter.

It may be that two different expressions may be found which represent the same number; in that case those two expressions would form an equation.

When there are several facts given in the statement of the problem, it will be found that some of the facts are used in forming the different abbreviations; when they are not all so used, the remaining facts, expressed in algebraic form by means of the abbreviations, will form the equations needed.

Model A.—A baby gets 9 steps away from her mother before the mother starts after it; the baby takes 7 steps while the mother takes 5, but 1 of the mother's steps is equal to 2 of the baby's. How many steps must the mother take to catch the baby?

Here, as in many problems, it pays to draw a simple diagram.

Let A represent the mother's position, B the place the baby gets to before the mother starts, and C the place where the mother catches it. The numbers that we need to speak of in explaining the problem would be

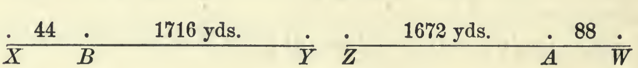
> the number of baby's steps from B to C; and
> the number of mother's steps from A to C.

If we let x represent either of these, an abbreviation for the other may be derived from the fact that "the baby takes 7 steps while the mother takes 5"; but for the answer asked for, it would be more convenient to let $x =$ the number of mother's steps from A to C. Then $\frac{7x}{5} =$ the number of baby's steps from B to C. Now we have the distance from A to B given equal to 9 steps, and that from B to C equal to $\frac{7x}{5}$ steps; but the distance from A to C, which is equal to x steps, is expressed in a different unit, namely, mother's steps; using the third fact of the problem, we can reduce the x mother-steps to $2x$ baby-steps, and we get the equation

$$2x = \frac{7x}{5} + 9.$$

Model B.—A and B run a mile. First A gives B a start of 44 yards, and beats him by 51 seconds; at the second heat A gives B a start of 1 minute 15 seconds, and is beaten by 88 yards. Find the time in which A and B can run a mile separately.

Drawing diagrams for the separate heats,



we see that in the first heat B ran only 1716 yards; in the second heat all the facts stated refer only to A's running 1672 yards. The numbers that we need to speak of in explaining the problem are

the number of seconds A takes to run a mile;
the number of seconds B takes to run a mile;
the number of seconds A takes to run 1672 yards;
the number of seconds B takes to run 1716 yards.

If we let x and y stand for the first two of these, we have:

$x =$ the number of seconds A takes to run a mile;

$y =$ the number of seconds B takes to run a mile;

$\frac{1672x}{1760} =$ the number of seconds A takes to run 1672 yards;

$\frac{1716y}{1760} =$ the number of seconds B takes to run 1716 yards.

Reducing these fractions to lowest terms we have

$$x + 51 = \frac{39y}{40}; \quad y + 75 = \frac{19x}{20}.$$

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601. The perimeter of a right triangle is 11 times as long as the shortest side. What is the ratio of the two sides containing the right angle?

602. Divide 33 into three parts so that the first may be to the second in the ratio 1.5, while the second is to the third as $\frac{1}{4}$ is to $\frac{1}{5}$.

603. The area of an oblong floor is 175 square feet, and its perimeter is 53 feet. Write down and solve the quadratic equation which gives the dimensions of the floor.

604. A vote was taken in a debating society, and the motion was carried, 5 to 3; on reconsideration, 50 of the affirmative votes deserted to the negative; if 60 more had deserted the motion would have been lost, 3 to 4. What was the vote on reconsideration?

605. A business man starts to walk to his house, a mile away; at the same moment his son starts from the house to run to the office and back. The son being able to run in 4 minutes a distance that the father walks in 9 minutes, where will he meet his father, and where will he overtake him?

606. The number of months in a man's age on his birthday in the year 1891 is exactly $\frac{1}{5}$ of the number denoting the year in which he was born. In what year was he born?

607. A man who can row a miles an hour in still water rows down a stream which flows b miles an hour and back to his starting-place. Show whether the time he takes is longer or shorter than the time required to row the same distance in still water.

608. An agent for a type-writing machine wishes to advertise that he will give one machine to any one selling a certain number. What must that number be in order that the agent may make an average profit of $5, when the machines cost the agent $20 and are retailed for $30?

609. A train goes from Oxford to London, 63 miles, 14 minutes faster than a train which travels 3 miles an hour slower. Speed of each train?

610. How much rice at 8 cents a pound must be mixed with 20 pounds at 11 cents in order that the mixture may be worth 10 cents a pound?

611. A fast express train makes its entire run at an average speed of $51\frac{1}{4}$ miles an hour; the up-grades at an average speed of 50 miles an hour, and the down-grades at an average speed of 54 miles an hour; there are 70 miles more up-grade than down-grade; what is the length of the run?

612. A man has 20 coins, of which some are half-dollars and the rest are nickels; if he should change the halves for dimes, and the nickels for cents, he would have 100 coins. How much money has he?

613. A miner bought 20 dogs, some at San Francisco and some at Juneau; they cost him on an average $10 apiece. If he had been able to get them all at San Francisco, he would have saved $160; if he had had to get them all at Juneau, he would have lost $160. How many were bought at each place?

614. A certain number exceeds twice the product of its two digits by 73, and 3 times the sum of its digits by 61; find the number.

615. Two trains start from opposite ends of a two-track road, and pass each other 6 hours later; one completes the run in 5 hours' less time than the other. Find the running time for each train.

616. Two pedestrians start together on a certain course, and one walks twice as fast as the other, over the whole course and back again; but the slower one walks only $\frac{5}{8}$ of the course and back again, being passed by the faster man $\frac{1}{2}$ mile from the winning-post. Find the length of the course.

617. The difference in the expense of fencing two square fields is $\frac{1}{4}$ of the difference in value of the fields themselves; and the total expense of fencing both fields is twice the difference in value of the fields. Supposing the cost of fencing per yard happens to be equal to the value of the land per square yard in either field, find the area of the fields.

618. A man buys 570 pulleys, some at 16 for a dollar and the rest at 18 for a dollar; he sells them all at 15 for a dollar and gains $3. How many of each sort does he buy?

619. The difference of two numbers multiplied by their product is 30, and the difference of their cubes is 98. Find the numbers.

620. My income is a third less than my brother's, but my expenses are 40 per cent. less than his; and we each save $100. Income of each?

621. A dozen pair of boots and a dozen pair of shoes can be had for $46; and a dozen pair more of shoes can be had for $105 than of boots for $100. Price of each per pair?

622. Find three numbers which are to each other as $2:4:5$, and such that the sum of the greatest and least exceeds the other by 21.

623. The sum of the squares of the two numbers formed

by the same pair of consecutive digits is 585. Find the numbers.

624. My brother and I can do a piece of work in 6 days; you and I would take 9 days to do it. In what time could you and he do it, supposing that he works twice as fast as I?

625. A sailor on shore leave spent $\frac{1}{3}$ of his wages in presents for his friends, $\frac{1}{4}$ for carriage-hire, $\frac{1}{5}$ for circus-tickets, had $\frac{1}{6}$ stolen, and retained \$1.50 when he returned to the ship. How much had he at first?

626. A large sum of money passes through the hands of an importing agent, a custom-house official, and a lawyer, each of whom takes from it a different percentage of the value it has when he receives it; if the sum remaining is \$1200, and the sum taken by the last of the three men is \$300, \$400, or \$600, according to the different orders in which the money can be successively submitted to them, what was the original sum?

627. A and B run a mile race. In the first heat B receives 4 seconds start, and is beaten by 32 yards; in the second heat B receives 64 yards start, and wins by 2 seconds. Find the time each takes to run a mile.

628. A and B worked together for 8 days, and completed half of a definite task that had been put before them; B then left the job, and A, after working with C, a new workman, for 6 days, left the job also; then C worked on alone, and completed the task in 2 more days. What C accomplished in the last two days was less than what A would have done by the exact amount that B would have done in one day. Find the time in which each workman could do the work alone, and the proportions in which they should be paid.

629. An egg-dealer bought a certain number of eggs at 32 cents per score, and 5 times that number at \$1.50 per

hundred. He sold them all at 20 cents per dozen, gaining \$648 by the transaction. How many eggs did he buy?

630. A number of soldiers drawn up in a hollow square 4 ranks deep can also be formed in a solid column of which the number of ranks will be equal to the number of men in each rank; the number of men in the front rank being 25 greater in one formation than in the other. Find the number of men.

631. A had $\frac{1}{4}$ as much money as B had; and since then he has paid B \$6. The money A now has bears to the sum he originally had the same ratio that the money B now has bears to \$15. How much had each at first? Explain the negative answer.

632. A man walks away from home a miles per hour, rests b hours, and then walks back at the rate of c miles per hour, being gone from home k hours in all. How far away did he get?

633. On what day of a year which is not a leap-year are the "days past" and the "days to come" of a calendar consecutive squares? Note that the "days past" for any given date include that day.

634. A letter-carrier is delayed half the time during the first 45 minutes of his trip of 6 miles; then he finds that he must go 2 miles per hour faster to get through on time. How many miles per hour is he expected to go?

635. At what time between 1 and 2 o'clock is the minute-hand as far from the hour-hand as the latter is from 12? Generalize this problem; discuss the solution for different hours, and the possibility of realizing the solutions on an actual clock.

636. A clock is set right at noon, January 1; when the correct time is 1:15 P.M. the same day, the hands of this clock are 6 minutes apart. When will it be an hour slow?

637. A certain number divided by another gives a quotient

3 and a remainder 2; if 9 times the second number be divided by the first, the quotient is 2 and the remainder 11. Find the numbers.

638. A number consists of two figures whose product is 21; if 22 be subtracted from the number, and the sum of the squares of its figures added to the remainder, the order of the figures will be inverted. What is the number?

639. A fraction becomes $\frac{3}{4}$ by the addition of 3 to the numerator and 1 to the denominator. If 1 be subtracted from the numerator and 3 from the denominator, the fraction becomes $\frac{1}{2}$. Find the fraction.

640. Divide 111 into three parts, such that the products of the several pairs may be in the ratios 4 : 5 : 6.

641. A and B run a race of 480 feet. The first heat A gives B a start of 48 feet, and beats him by 6 seconds; the second heat A gives B a start of 48 yards, and is beaten by 2 seconds. How many feet can each run in a second?

642. A and B set out at the same time to walk to a place 6 miles distant and back again. After walking for 2 hours, A meets B coming back. Supposing B to walk twice as fast as A, find their respective rates of walking.

643. A man bought a certain number of eggs for $2. If he had paid 5 cents more per dozen, he would have received 2 dozen less for the same money. How many dozen did he buy, and what did he pay for them?

644. A boy spent his money in oranges. If he had bought 5 more, each orange would have cost him $\frac{1}{2}$ cent less; if 3 less, $\frac{1}{2}$ cent more. How much did he spend, and how many did he buy?

645. A number is made up of three figures whose sum is 17. The figure of the hundreds is double that of the units. When 396 is subtracted, the order of the figures is reversed. What is the number?

646. The smaller of two numbers divided by the larger is

.21, with a remainder .04162; the greater divided by the smaller is 4, with .742 for a remainder. What are the numbers?

647. A number is made up of three figures whose sum is 17. The figure of the units is $\frac{2}{3}$ that of the hundreds. When 297 is subtracted, the order of the figures is inverted. What is the number?

648. A and B together can do $\frac{1}{4}$ of a piece of work in 6 days. If B can do $\frac{1}{3}$ of it in 18 days, how long will it take A to do $\frac{1}{3}$ of it?

649. A person sets out at the rate of 11 miles in 5 hours; 8 hours after, another person sets out from the same place, and goes after him at the rate of 13 miles in 3 hours. How far must the latter travel to overtake the former?

650. A certain number of persons were divided into three classes, such that the majority of the first and second together over the third was 10 less than 4 times the majority of the second and third together over the first; but if the first had 30 more, and the second and third together 29 less, the first would have outnumbered the last two by 1. The whole number was 34 more than 8 times the majority of the third over the second. Find the number in each class.

651. A person has a certain sum, half of which he lends at 5 per cent interest, and half at 4^1 per cent interest. The first loan yields him $60 more interest than the other. What is the amount of his capital?

652. From the two formulas in simple interest, $a = p + i$ and $i = trp$, find what p equals in terms of a, t, and r. Express in words the truth you have expressed in letters. Make an application of this truth in solving a problem with numbers in simple interest.

653. The width of a rectangular garden is 2 rods less than its length, and $\frac{3}{5}$ of its area is equivalent to 8 rods

less than $\frac{7}{8}$ of the square of its width. How many rods long and wide is the garden?

654. If the side of a square were 5360 centimeters longer it would contain 65536 square metres. What is the length of a side?

655. Thirty feet more than $\frac{1}{9}$ of the present height of the highest of the Egyptian pyramids is equal to $\frac{1}{6}$ of its original height. Three hundred feet less than 6 times its present height is equal to 5 times its original height. Find the original and the present height.

656. One of the parallel sides of a trapezoid is 3 inches longer than the other; and the altitude is 1 inch greater than the shorter of these. The area is 68 square inches. Find the altitude and the length of each base.

657. The product of the sum and difference of two numbers is a, and the product of the sum of their squares by the difference of their squares is ma. Find the numbers.

658. How long will it be before the hands of the clock will again assume the same relative position that they have at this moment?

659. On a certain street railway two sizes of cars are used. What is the seating capacity of each, if 14 more persons can be seated in 3 large cars than in 4 small ones, and 2 more persons in 2 large cars than in 3 small ones?

660. A rectangular lawn 20m. 5 dm. long and 8m. 5dm. wide has a path of uniform width around it. If the area of the path equals 62 ca., what is its width?

661. A takes 3 hours longer than B to walk 30 miles; but if he doubles his pace he takes 2 hours less time than B. Find their respective rates of walking.

662. Divide 20 into three parts such that the products of the three pairs may be in the ratios 6 : 10 : 15.

663. The first digit of a number is 3 times the second; and

if the number, increased by 3, be divided by the difference of its digits, the quotient is 16. Required the number.

664. Find the number whose cube root is $\frac{1}{5}$ of its square root.

665. A and B can do a piece of work together in 8 days. A works alone 4 days, and then both finish it in 5 days more. In what time could each have done it alone?

666. The sum of two numbers is 16, and the sum of their reciprocals is $\frac{1}{3}$. What are the numbers?

667. Find two numbers such that their product, their sum, and the difference of their squares shall be equal to each other.

668. A certain number of two digits is equal to twice the sum of its digits, and the number got by interchanging the digits is equal to the sauare of the sum of the digits. Find the number.

669. Find three numbers of which the first is greater than the second by as many units as the second is greater than the third · the sum of the squares of the three being 66.

670. Find two numbers whose product is 78, such that if one be divided by the other the quotient is 2 and the remainder 1.

671. Divide 1152 into three parts, such that 9 times the sum of the first and second shall be equal to 7 times the sum of the second and third; and if 8 times the first be subtracted from 8 times the second, the remainder shall be equal to the sum of the first and third.

672. Find the number whose square added to its cube is 9 times the next higher number.

673. Two travellers set out from two distant towns and go towards each other. When they meet they find that one of them has gone 30 miles more than the other, and can complete his journey to the other town in 4 days, while the

other will need 9 days to complete his journey. How far apart are those two towns?

674. A cask contains 12 gallons of wine and 18 gallons of water, and another cask contains 9 gallons of wine and 3 gallons of water; how many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water?

675. A deer fleeing from a tiger gets 60 jumps away from a certain tree before the tiger passes that tree; she makes 6 jumps while the tiger makes 5, but 7 of the tiger's jumps are equal to 9 of the deer's. How many jumps from the tree will the tiger go before he catches the deer?

676. Two workmen were employed at different wages and paid at the end of a certain time. The first, who had worked all the time, received \$26.25, and the second, who worked 6 days less, received \$19.80. If the second had worked all the time, and the first had omitted six days, their wages, taken together, would have amounted to \$48.75. How many days did each work, and what were the wages of each?

677. A dealer sells two kinds of goods, 8 yards more of the second kind than of the first, and receives \$100 from the sale. He then finds that he has left just as much of the first kind as he has sold of the second, and that the quantity of the first kind thus left is worth \$100; and that he has left of the second kind just as much as he has sold of the first, and that the quantity left of the second kind is worth \$16. Find the number of yards sold of each kind, and the price of each per yard.

678. Two horsemen start at the same time, on the same road, from two places 15 miles apart. At the end of 10 hours the second horseman overtakes the first, and on comparing their rates they find that there has been a difference

of 5 minutes in the time of going every 7 miles. Find their rates, and the distances they have gone.

679. Two pipes which supply the same reservoir fill it in 4 hours and 12 minutes when both run together; but the first pipe alone can fill it in one hour less than half the time in which the second pipe alone can fill it. Find the time for each pipe alone.

680. Find a number such that if it be multiplied by 4 and the product increased by 3, the result shall be the same as if it were increased by 4 and the sum multiplied by 3.

681. Compare the rates of two boat's crews, one rowing 10 miles down-stream in an hour, the other 5 miles down and 5 miles back again in an hour; supposing the current to run 2 miles an hour.

682. A sets off from Boston to New York, and B at the same time from New York to Boston, and they travel uniformly; after they have met on the way, it takes A 16 hours to reach New York, and B 36 hours to reach Boston. Find in what time each performed the journey.

683. A man hires a certain number of acres of land for \$336. He cultivates 7 acres for himself, and, by letting the rest for \$4 an acre more than he paid for it, gets his own patch rent free. Find the number of acres he hired.

684. A laborer having built 105 rods of stone wall found that if he had built 2 rods less a day he would have been 6 days longer in completing the job. How many rods a day did he build?

685. A and B have together $\frac{2}{3}$ as much money as C; B and C have together 6 times as much as A; and B has \$680 less than A and C have together. How much has each?

686. A certain man has \$1400, which he separates into two portions, and puts at interest at different rates, but so that the two portions produce equal returns. If the first portion had been lent at the second rate of interest, it would

have brought in \$18 per annum; and if the second portion had been lent at the first rate, it would have brought in \$32 per annum. Find the rates of interest and the two portions of the principal.

687. On a certain road the telegraph-poles are placed at equal intervals, and their number is such that if that number were less by one, each interval between two poles would be increased by $2\frac{14}{15}$ yards. Find the number of poles and the number of intervals in a mile.

688. Two workmen, A and B, working together on the same job, complete it in 15 days. It is found, on the comparison of their work, that A, if working alone, would have required 16 more days for doing the whole job than B would have needed for it, working alone. Find the time in which each workman would have done the job if working alone.

689. A certain share of the profits of a farm is divided equally every month among the hands. In July the number of hands was 8 more than in June, and each hand received \$6 less; the sum divided being \$700. In August the number of hands was 2 less than in June, and each hand received \$1 less; the sum divided being \$540. Find the sum divided in June.

690. A certain cistern is supplied by three pipes, A, B, and C, of which A, when running, discharges into the cistern 1000 gallons per hour. Pipe B, running alone, would fill the cistern in 8 hours less than A alone; pipe C, running alone, would require 5 hours more than A alone; while A and B, running together, would fill the cistern in $\frac{3}{10}$ the time required by C alone. Find the number of gallons which the cistern is capable of holding.

691. Two men, A and B, had a money-box, containing \$210, from which each drew a certain sum daily; this sum being fixed for each, but different for the two. After six weeks the box was empty. Find the sum which each man

drew daily from the box; knowing that A alone would have emptied it 5 weeks earlier than B alone.

692. A party of friends went on a pleasure excursion, the expense of which they shared equally. If the number of the party had been decreased by 7, and if the total expense had been \$150, the assessment for each person would have been \$1 more than it was; but if the number of the party had been increased by 8, and if the total expense had been \$160, the assessment for each person would have been \$1 less than it was. Find the number of the party, the assessment for each person, and the total expense of the excursion.

693. A battalion of soldiers, when formed into a solid square, present 16 men fewer in the front than they do when formed in a hollow square 4 deep. How many men are there in the battalion?

694. Two mowers, working steadily together, mow a certain field in 3 hours. On comparing their several shares of the work done, they find that A would have required, to do B's share, $2\frac{1}{2}$ hours more than B would have required to do A's share. Find the time in which each mower, working alone, would have mowed the whole field.

695. Two workmen, A and B, are employed on a certain job at different wages. When the job is finished, A receives \$27, and B, who has worked 3 days less, receives \$18.75. If B had worked for the whole time, and A 3 days less than the whole time, they would have been entitled to equal amounts. Find the number of days each has worked, and the pay each receives per diem.

696. Several friends on an excursion spent a certain sum of money. If there had been 5 more persons in the party, and each person had spent 25 cents more, the bill would have amounted to \$33. If there had been 2 less in the party, and each person had spent 30 cents less, the bill

would have amounted to only \$11. Of how many did the party consist, and what did they spend?

697. A man walks 2 hours at the rate of $4\frac{1}{2}$ miles per hour; he then adopts a different rate. At the end of a certain time he finds that if he had kept on at the rate at which he had set out, he would have gone three miles further from his starting-point; and that if he had walked 3 hours at his first rate and $\frac{1}{2}$ hour at his second rate, he would have reached the point he has actually attained. Find the whole time occupied by the walk, and his final distance from the starting-point.

698. A reservoir, supplied by several pipes, can be filled in 15 hours, every pipe discharging into it the same fixed number of hhds. per hour. If there were 5 more pipes, and every pipe discharged per hour 7 hhds. less, the reservoir would be filled in 12 hours. If the number of pipes were 1 less, and every pipe discharged per hour 8 hhds. more, the reservoir would be filled in 14 hours. Find number of pipes and capacity of reservoir.

699. A man bought a certain number of railway shares when they were at a certain rate per cent discount for \$8500; and afterwards, when they were at the same rate per cent premium, he sold all but 20 of them for \$9200. How many did he buy, and what did he give for each of them?

700. A and B can do a piece of work in 18 days; A and C can do it in 45 days; B and C in 20 days. Find the time in which A, B, and C can do it, working together.

701. A man bought a certain number of sheep for \$300; he kept 15 sheep, and sold the remainder for \$270, gaining half a dollar a head. How many sheep did he buy, and at what price?

702. A hires a certain number of acres for \$420. He lets all but 4 of them to B, receiving for each acre \$2.50

more than he paid for it. The whole amount received from B is $420. Find the number of acres.

703. A man walks at a regular rate of speed on a road which passes over a certain bridge distant 21 miles from the point the man has reached at noon. If his rate of speed were half a mile per hour greater than it is, the time at which he crosses the bridge would be an hour earlier than it is. Find his actual rate of speed and the time at which he crosses the bridge Explain the negative answer.

704. A man setting out on a journey drives at the rate of a miles per hour to the nearest railway station, distant b miles from his house. On arriving at the station he finds that the express for his destination has left c hours before. At what rate should he have driven to catch the express? Having obtained the general solution, find what the answer becomes in the following cases:

$$(1)\ c = 0; \quad (2)\ c = \frac{b}{a}; \quad (3)\ c = -\frac{b}{a}.$$

In case (2) how much time does the man have to drive from his house? In case (3) what is the meaning of the negative value of c?

705. A landowner laid out a rectangular lot containing 1200 square yards. He afterwards added 3 yards to one dimension of his lot, and subtracted $1\frac{1}{2}$ yards from the other, thereby increasing the area of his lot by 60 square yards. Find the dimensions of the lot before and after the change. How do you explain the negative answer?

706. A vessel is half full of a mixture of wine and water. If filled up with wine, the ratio of the quantity of wine to the quantity of water is 10 times what it would be if the vessel were filled up with water. Find the ratio of the original quantity of wine to that of water.

707. Three students, A, B, and C, agree to work out a

series of difficult problems, in preparation for an examination; and each student determines to solve a fixed number every day. A solves 9 problems per day, and finishes the series 4 days before B; B solves 2 more problems per day than C, and finishes the series 6 days before C. Find the number of problems, and the number of days given to them by each student.

708. A certain whole number composed of three digits has the following properties: 10 times the middle digit exceeds the square of half the sum of the digits by 21; if 99 be added to the number, the order of the digits is inverted; and if the number be divided by 11, the quotient is a whole number of two digits, which are the same as the first and last digits of the original number. Find the number.

709. A certain manuscript is divided between A and B to be copied. At A's rate of work, he would copy the whole manuscript in 18 hours; B copies 9 pages per hour. A finishes his portion in as many hours as he copies pages per hour; B is occupied 2 hours more than A upon his portion. Find the number of pages in the manuscript, and the number of pages in the two portions.

710. Two casks, of which the capacities are in the ratio of a to b, are filled with mixtures of water and alcohol. If the ratio of water to alcohol is that of m to n in the first cask, and that of p to q in the second cask, what will be the ratio of water to alcohol in a mixture composed of the whole contents of the two casks? Reduce the answer to its simplest form. What does the answer (in its simplest form) become if $m = q = 0$; and what is the simplest statement of the question in this case?

711. A boat's crew, rowing at half their usual speed, row 3 miles down a certain river and back again, in the middle of the stream, accomplishing the whole distance in 2 hours

and 40 minutes. Find (in miles per hour) the rate of the crew when rowing at full speed, and the rate of the current. (Notice *both* solutions of this problem.)

712. A and B have 4800 circulars to stamp for the mail, and mean to do them in 2 days, 2400 each day. The first day, A, working alone, stamps 800 circulars, and then A and B together stamp the remaining 1600; the whole job occupying 3 hours. The second day A works 3 hours, and B 1 hour; but they accomplish only $\frac{9}{10}$ of their task for that day. Find the number of circulars which each stamps per minute, and the length of time that B works on the first day.

713. A broker sells certain railway shares for \$3240. A few days later, the price having fallen \$9 per share, he buys for the same sum 5 more shares than he had sold. Find the price, and the number of shares transferred each day.

714. At 6 o'clock on a certain morning, A and B set out on their bicycles from the same place, A going north and B south, to ride until 1:30 P.M. A moved constantly northwards at the rate of 6 miles per hour. B also moved always at a fixed rate; but after a while he turned back to join A. Four hours after he turned, B passed the point at which A was when B turned; and at 1:30 P.M., when he stopped, he had reduced, by one-half, the distance that was between them at the time of turning. Find B's rate, the time at which he turned, the distance between A and B at that time, and the time at which B would have joined A if the ride had been continued at the same rates of speed. Find the answers for *both solutions.*

715. Two travellers, A and B, go from P to Q at uniform but unequal rates of speed. A sets out first, travelling on foot at the rate of 20 minutes for every mile. B follows, going 1 mile while A traverses the distance $\frac{PQ}{80}$. B

overtakes and passes A, 8 miles from P; and when B reaches Q, he is 9 miles ahead of A. Find the distance PQ, and B's rate of speed in minutes to the mile.

716. Tristram is 10 years younger than Launcelot; and the product of the ages they attained in 1870 is 96. Find the ages they attain in 1888.

717. A certain librarian spends every year a fixed sum for books. In 1886, the cost of his purchases averaged $2 per volume; in 1887, he bought 300 more volumes than in 1886; and in 1888, 300 more volumes than in 1887. The average cost per volume was 30 cents lower in 1888 than in 1887. Find the number of volumes bought each year, and the fixed price paid for them.

718. Two tanks, A and B, are discharging water; A at the rate of x barrels per hour, and B at the rate of $x + 100$ barrels per hour. At a time $(1 + y)$ hours after noon, A contains 480 barrels less than at noon; and at a time $(1 - y)$ hours after noon, B contains 400 barrels less than at noon. Find the rate at which each tank is discharging water; and the times $(1 + y)$ and $(1 - y)$ hours after noon.

719. A certain railway runs due east and west, P, Q and R being successive stations on the road from east to west, and so situated that PQ = 216 miles, PR = 240 miles. Two trains, on parallel tracks, pass P simultaneously at noon. The rate of motion of train No. 2 from east to west is 8 miles per hour less than that of train No. 1; and train No. 2 passes Q $1\frac{1}{2}$ hours later than train No. 1 passes R. Find the rate and direction of motion of each train; and find the hour at which train No. 1 passes R, and the hour at which train No. 2 passes Q.

720. Two wheelmen, A and B, are riding eastward over the same road, B's eastward rate exceeding A's by 4 miles an hour. A certain milestone, M, is passed by A at noon, and by B 15 minutes later; a second milestone, N, 6 miles

east of M, is passed by both wheelmen at the same instant. Find the rate of each wheelman in miles per hour, and the time of their passing N.

721. A and B start at the same time from two towns and travel towards each other. When they meet, B has travelled a miles more than A; and it will take A b hours longer, and B c hours longer, for each to reach the town the other has left. Find the distance between the towns.

722. A gentleman has two horses and one chaise. The first horse is worth a dollars less, and the second horse b dollars less, than the chaise. If $\frac{3}{5}$ of the value of the first horse be subtracted from that of the chaise, the remainder will be the same as if $\frac{7}{3}$ of the value of the second horse is subtracted from twice that of the chaise. Find the value of each horse and that of the chaise. What are the answers if $a = -50$; $b = 50$?

723. Divide the number a into two such parts that if the first be multiplied by m and the second by n, the sum of the products is b. In what case would the terms of the fractional values of the unknown quantities become zero? and how could they, then, satisfy the conditions of the problem?

724. A courier started from a certain place n days ago, and makes a miles daily. He is pursued to-day by another making b miles daily. In how many days will the second overtake the first? In what case would both the terms of the fractional value of the unknown quantity become zero? and how could this value be a solution?

725. A wine merchant has two kinds of wine: the one costs a shillings per gallon, the other b shillings. How must he mix both these wines together in order to have n gallons worth c shillings per gallon? In what cases would the values of either of the unknown quantities be negative? Why should this be the case, and could the enun-

ciation be corrected for this case? In what cases would the value of one of the unknown quantities be zero, and what would this value signify?

726. The owners of a certain mill make a dollars a day each, sharing equally. If the number of owners were b less, they would make c dollars each. Required the number of owners and the total daily profit. What are the answers if $a = 80$, $b = -3$, $c = 50$?

727. Two trains are running westward on parallel tracks, at the rate of a and b miles an hour respectively. At noon they are m miles apart. When are they together? When is the unknown quantity positive? When does it become zero? infinite? indeterminate? negative? Explain the signification of each result.

728. A certain sum of money will amount to a dollars in m months, and to b dollars in n months. Find the principal and the rate of interest. Find the answers when $a = 1837.50$, $b = 1890$, $m = 10$, $n = 16$.

729. A banker has two kinds of coin. It takes a pieces of the first, or b pieces of the second, to make a dollar. If a dollar is offered for c pieces, how many of each kind must be given?

730. I have a counters distributed unequally among three cups. Taking some of the counters from the first cup, I double the number of counters in the second and third; next, taking from the second, I double the number in the first and third; and lastly, taking from the third cup, I double the numbers in the first and second. Then I find that the second cup contains twice, and the third cup three times, as many counters as the first. How many in each cup at the beginning?

731. A merchant who had two brands of flour sold a barrels of the first and b barrels of the second at an average price of \$$c$ per barrel; and at the same rates, he sold m

barrels of the first and n barrels of the second at an average price of \$$p$ per barrel. Find the price of each brand.

732. A person has a hours at his disposal. How far can he ride in a coach at the rate of b miles an hour, so as to get back in time, walking at the rate of c miles an hour?

733. A cask of wine contains a gallons; b gallons are drawn off, and the cask filled up with water. After this has been done n times, how many gallons of the original wine are in the cask?

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